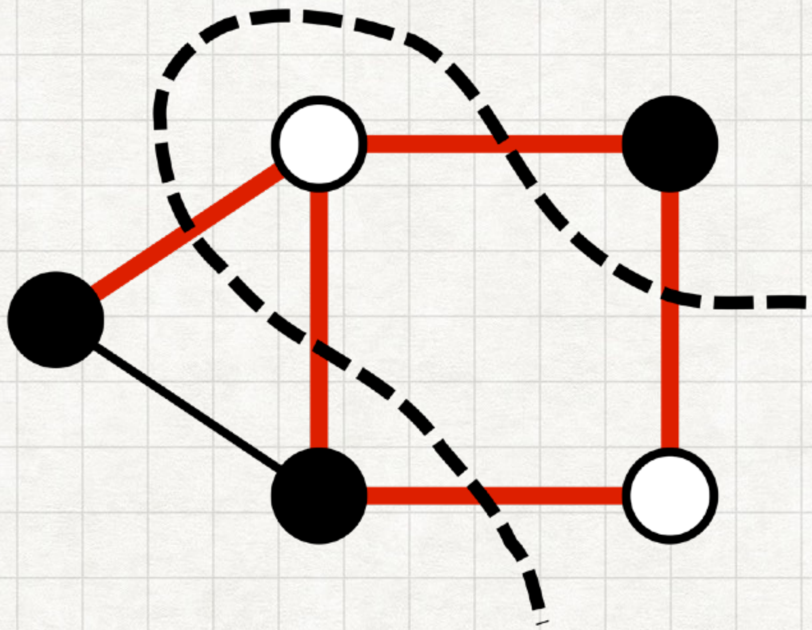
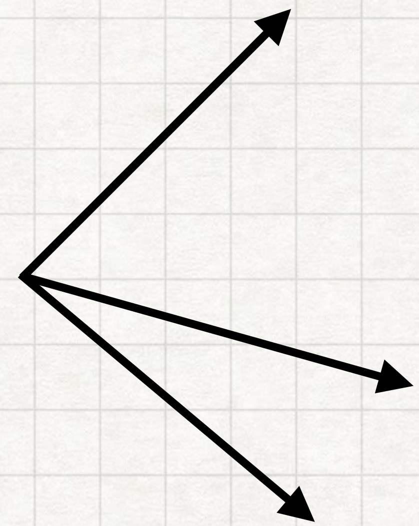
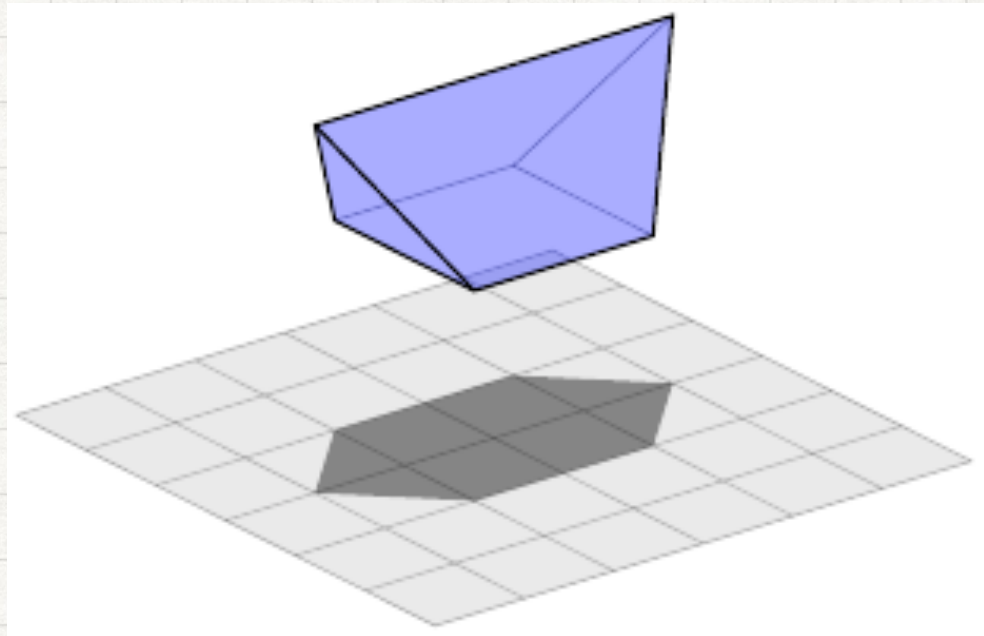






MAXCUT, ORTHONORMAL REPRESENTATIONS, AND EXTENSION COMPLEXITY OF POLYTOPES

By: Igor Balla

Question(Lovász): What is the maximum of the length $||\sum_{i=1}^{n} x_i||$ over all unit vectors $x_1, \ldots, x_n \in \mathbb{R}^d$ such that among any three, some pair is orthogonal?

Answer(Konyagin 81, Alon 94): $\Theta(n^{2/3})$.

Question(Erdős): What is the maximum number of vectors in $\mathbb{R}^d$ such that among any three, some pair is orthogonal?

Answer(Rosenfeld 81): $2d$.

Question(Erdős): What about if among any $k+1$, some pair is orthogonal? Is the answer kd?

Answer and follow-up question(Füredi and Stanley 92): No! But maybe the answer is still at most $(kd)^{O(1)}$?

Answer(Alon and Szegedy 99): No! Can have $d^{\Omega(\log k / \log\log k)}$ many vectors.

Nearly Equivalent Questions

Setup: Let R be a Euclidean space with inner product $\langle \cdot, \cdot \rangle$. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$.

Definition: $f\colon V(G) \to R$ is called an *orthonormal representation* if

- $f(v)$ is a unit vector,
- $\langle f(u), f(v) \rangle = 0$ for distinct $u, v \in V(G)$ such that $uv \notin E(G)$.

Definition: The *minimum semidefinite rank* $\mathrm{msr}(G)$ is the minimum d such that there exists an orthonormal representation $f\colon V(G) \to \mathbb{R}^d$.

Question(B., Letzter, Sudakov 20): What is the minimum of $\mathrm{msr}(G)$ over all H-free graphs G with n vertices?

Answer(Rosenfeld 81): $\lceil n/2 \rceil$ when H is a triangle.

Nearly Equivalent Questions

Question(B., Letzter, Sudakov 20): What is the minimum of $\mathrm{msr}(G)$ over all H-free graphs G with n vertices?

Answer(B., Letzter, Sudakov 20): $\lceil n/k \rceil$ when H is a cycle of length $k+1$.

Answer(Alon and Szegedy 99): At most $n^{O(\log\log k/\log k)}$ when H is the complete graph on k vertices.

Answer(B. 24): At most $n^{O(\log\log k/\log k)}$ when H is the complete bipartite graph with parts of size k.

Cool, but why should we care?

Nearly Equivalent Questions

Definition: The *Lovász theta function* $\vartheta(G)$ is the maximum over all orthonormal representations f of $\overline{G}$, of the largest eigenvalue of the Gram matrix defined by $M(u,v) = \langle f(u), f(v) \rangle$

Question(B., Letzter, Sudakov 20): What is the maximum of $\vartheta(\overline{G})$ over all H-free graphs G with n vertices?

Answer(Konyagin 81, Alon 94): $\Theta(n^{1/3})$ when H is a triangle.

Answer(B., Letzter, Sudakov 20): At most $O(n^{1/k})$ when H is the cycle of length k, for a fixed k.

Tight if k is odd or if k is 4, 6, 10.

Due to "optimally pseudorandom" H-free graphs which are extremal for the bipartite Turán number problem.

Nearly Equivalent Questions

Question(B., Letzter, Sudakov 20): What is the maximum of $\vartheta(\overline{G})$ over all H-free graphs G with n vertices?

Answer(Feige 95): At least $n^{1-O(1/\log k)}$ when H is the complete graph on k vertices.

When H is a complete bipartite graph, an ``optimally pseudorandom" H-free graph can at best give $\vartheta(\overline{G}) \geq \Omega(\sqrt{n})$.

Question(B., Letzter, Sudakov 20): Is the answer closer to $\sqrt{n}$ or n for H being a complete bipartite graph with parts of size k?

Answer(B. 24) At least $n^{1-O(\log\log k/\log k)}$.

Theorem(Lovász 79): $\vartheta(G) \leq \mathrm{msr}(G)$ and $\vartheta(G)\vartheta(\overline{G}) \geq n$

So $\vartheta(\overline{G}) \geq n/\mathrm{msr}(G)$.

MaxCut

Definition: For a graph G, $\mathrm{MaxCut}(G)$ is defined to be the maximum over all partitions A, B of the vertex set $V(G)$ of the number of edges between A and B.

Theorem(Edwards 73): $\mathrm{MaxCut}(G) \geq m/2 + \Omega(\sqrt{m})$ for any graph G with m edges.

Question(Alon, Bollobás, Krivelevich, Sudakov 03): What is the minimum of $\mathrm{MaxCut}(G) - m/2$ over all H-free graphs G with m edges?

Erdos and Lovász 79, Shearer 92 studied the case H a triangle.

Answer(Alon 94, 96): $\Theta(m^{4/5})$ when H is the triangle.

Conjecture(Alon, Bollobás, Krivelevich, Sudakov 03): For any fixed H, there exists $\varepsilon > 0$, such that the answer is at least $\Omega(m^{3/4+\varepsilon})$.

MaxCut

Question(Alon, Bollobás, Krivelevich, Sudakov 03): What is the minimum of $\mathrm{MaxCut}(G) - m/2$ over all H-free graphs G with m edges?

Answer(Alon, Krivelevich, Sudakov 05): $\Omega(m^{(k+1)/(k+2)})$ when H is an even cycle of length k.

Answer(Glock, Janzer, and Sudakov 23): $\Omega(m^{(k+1)/(k+2)})$ when H is an odd cycle of length k.

Theorem(B., Janzer, and Sudakov 24):
$\mathrm{MaxCut}(G) - \frac{m}{2} \geq \frac{1}{\pi} \cdot \frac{m}{\vartheta(\overline{G})-1}$ for any graph G with m edges.

Corollary(B., Janzer, and Sudakov 24): $\Omega(m^{(k+1)/(k+2)})$ when H is a cycle of length k.

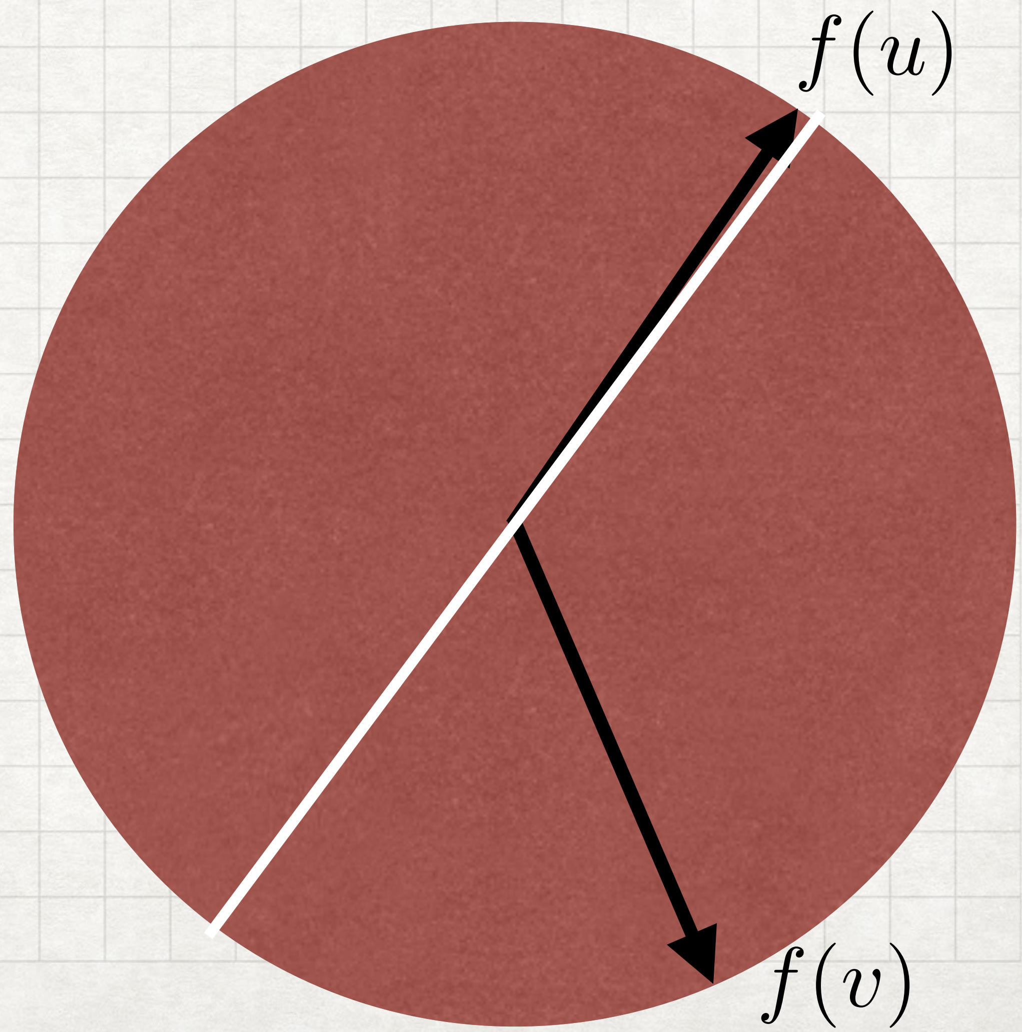
MaxCut

Theorem(B., Janzer, and Sudakov 24):

$\mathrm{MaxCut}(G) - \frac{m}{2} \geq \frac{1}{\pi} \cdot \frac{m}{\vartheta(\overline{G})-1}$ for any graph G with m edges.

Definition: The *Lovász theta function* $\vartheta(G)$ is the minimum κ, such that there exists a map $f : V(G) \to (\text{unit sphere})$ satisfying $\langle f(u), f(v) \rangle = -1/(\kappa - 1)$ for all $uv \notin E(G)$.

Proof idea: Choose a random hyperplane and partition the vertices according to which side of it they fall on.



The probability that edge uv falls in different sides is

$$\frac{1}{2} + \frac{1}{\pi} \arcsin\left(-\langle f(u), f(v) \rangle\right)$$
$$\geq \frac{1}{2} + \frac{-\langle f(u), f(v) \rangle}{\pi}$$

MaxCut

Definition: The *vector chromatic number* $\chi_{\text{vec}}(G)$ is the minimum κ, such that there exists a map $f : V(G) \to (\text{unit sphere})$ satisfying $\langle f(u), f(v) \rangle \leq -1/(\kappa - 1)$ for all $uv \in E(G)$.

Theorem(B., Janzer, and Sudakov 24):
$\text{MaxCut}(G) - \frac{m}{2} \geq \frac{1}{\pi} \cdot \frac{m}{\chi_{\text{vec}}(G)-1}$ for any graph G with m edges.

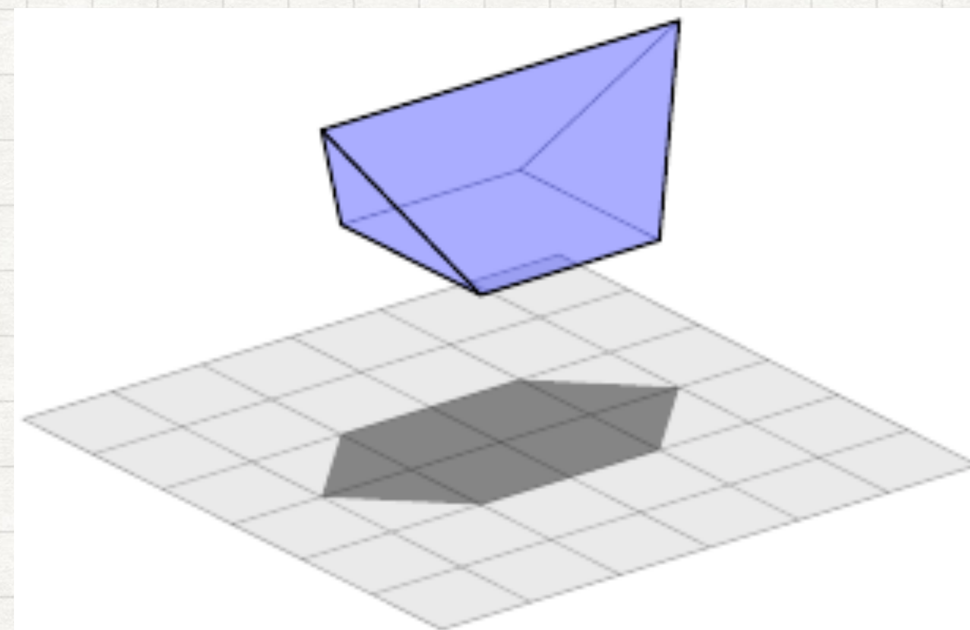
Theorem(B. 24): $\chi_{\text{vec}}(G) \geq \frac{\vartheta(\overline{G})^2}{n}$ and $\chi_{\text{vec}}(G) \geq \frac{n}{\text{msr}(G)}$.

Theorem(B. 24): There exists an H-free graph G with m edges satisfying $\chi_{\text{vec}}(G) \geq m^{1/2-O(\log\log k/\log k)}$, where H is the complete bipartite graph with parts of size k .

Conjecture(Elphick 23): $\text{MaxCut}(G) - \frac{m}{2} \geq \frac{1}{3} \cdot \frac{m}{\chi_{\text{vec}}(G)-1}$.

Extension Complexity of Polytopes

Definition: For a d-dimensional polytope P, its *extension complexity* $\mathrm{xc}(P)$, is the minimum number of facets of a polytope P' such that P is the orthogonal projection of P' onto some d-dimensional subspace.



Theorem(Kwan, Sauermann, Zhao 22): There exists an $n^{o(1)}$-dimensional polytope with at most n vertices and extension complexity at least $n^{1-o(1)}$.

Extension Complexity of Polytopes

Any polytope can be described by constraints:

$$P = \{v : b_i - \langle a_i, v \rangle \geq 0, i = 1, \ldots, t\}$$

slack

One can define a corresponding *slack matrix* whose entries are the slacks of the vertices of P.

Definition: For a nonnegative matrix M, its *nonnegative rank* $\mathrm{rank}_+(M)$ *is* defined to be the minimum r such that we can write $M = \sum_{i=1}^r R_i$, where each R_i is nonnegative and rank 1.

Theorem(Yannakakis 91): The extension complexity of any polytope P equals the nonnegative rank of any slack matrix of P.

Question(Hrubeš 12): What is the maximum of $\frac{\mathrm{rank}_+(M)}{\mathrm{rank}(M)}$ over all nonnegative $n \times n$ matrices?

Extension Complexity of Polytopes

Question(Hrubeš 12): What is the maximum of $\frac{\mathrm{rank}_+(M)}{\mathrm{rank}(M)}$ over all nonnegative $n \times n$ matrices?

Answer(Kwan, Sauermann, Zhao 22): At least $n^{1-O\left(\log\log n/\sqrt{\log n}\right)}$

Answer(B. 24): At least $n^{1-O\left(\sqrt{\log\log n}/\sqrt{\log n}\right)}$.

Proof sketch: For a nonnegative matrix M, $\mathrm{rank}_+(M)$ is at least the minimum number of rectangles whose union is the support of M.

Consider the Gram matrix M of an orthonormal representation of an H-free graph on n vertices (for H being the complete bipartite graph with parts of size k).

So any *rectangle* in the support of M has cardinality at most $2kn$ and thus $\mathrm{rank}_+(M) \geq \frac{m}{2kn}$, where m is the size of the support.

Can the following be improved (or proven)?

Let $\mathcal{F}_n$ be the family of all H-free graphs on n vertices where H is the clique of size k.

$$\Omega\left(n^{3/k}\right) \leq \min\{\mathrm{msr}(G) : G \in \mathcal{F}_n\} \leq n^{O(\log\log k/\log k)}$$

$$n^{1-O(1/\log k)} \leq \max\{\vartheta(\overline{G}) : G \in \mathcal{F}_n\} \leq O\left(n^{1-2/k}\right)$$

Fix H and let $\mathcal{F}_m$ be the family of all H-free graphs with m edges.

Conjecture(Alon, Krivelevich, Sudakov 05): Does there exist $c(H)$ such that $\min\{\mathrm{MaxCut}(G) - m/2 : G \in \mathcal{F}_m\} = \Theta\left(m^{c(H)}\right)$?

Conjecture(Alon, Bollobás, Krivelevich, Sudakov 03): $c(H) > 3/4$.

Conjecture(Elphick 23): $\mathrm{MaxCut}(G) - \frac{m}{2} \geq \frac{1}{3} \cdot \frac{m}{\chi_{\mathrm{vec}}(G)-1}$.

$$n^{1-O\left(\sqrt{\frac{\log\log n}{\log n}}\right)} \leq \max\left\{\frac{\mathrm{rank}_+(M)}{\mathrm{rank}(M)} : M \text{ is nonnegative and } n \times n\right\} \leq ?$$


thank you
danke
謝謝
ngiyabonga
teşekkür ederim
gracias
merci
obrigado
bedankt
dziękuję
go raibh maith agat
mochchakkeram
sukriya
kop khun krap
terima kasih
grazie
arigatô
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