RAMSEY GOODNESS OF BOUNDED DEGREE TREES

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Joint work with: Alexey Pokrovskiy, Benny Sudakov

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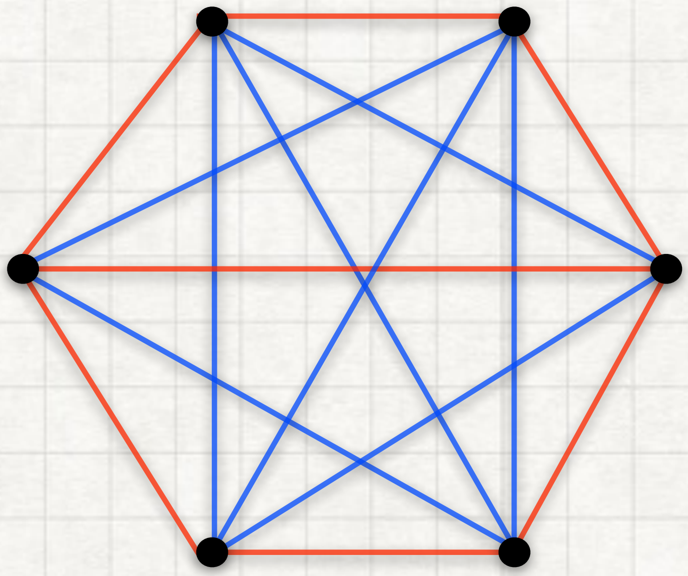

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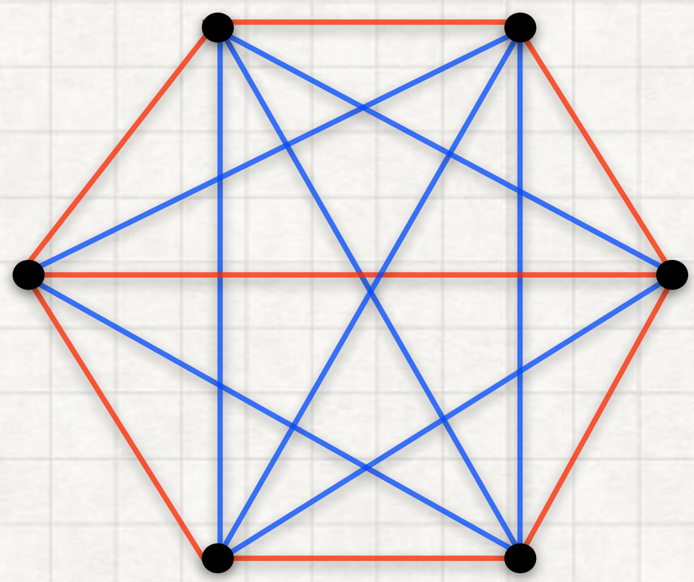

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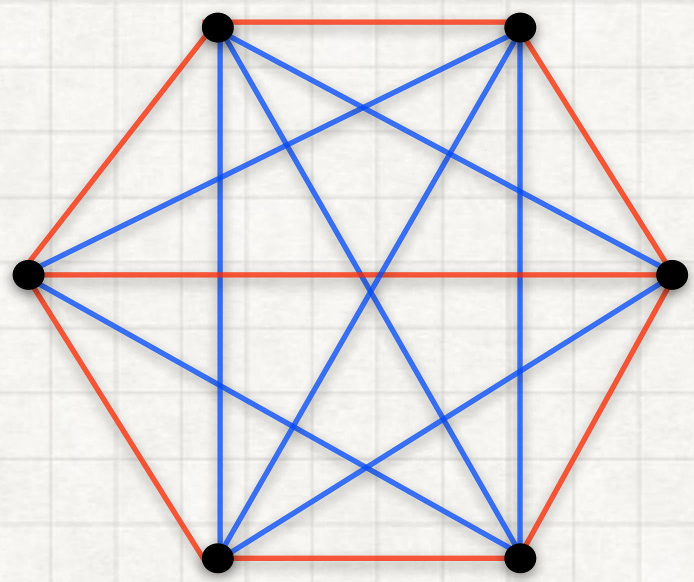

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Theorem (Erdős 1947; Erdős and Szekeres 1935):

$$\left(\sqrt{2}\right)^n \le R(K_n,K_n) \le 4^n$$

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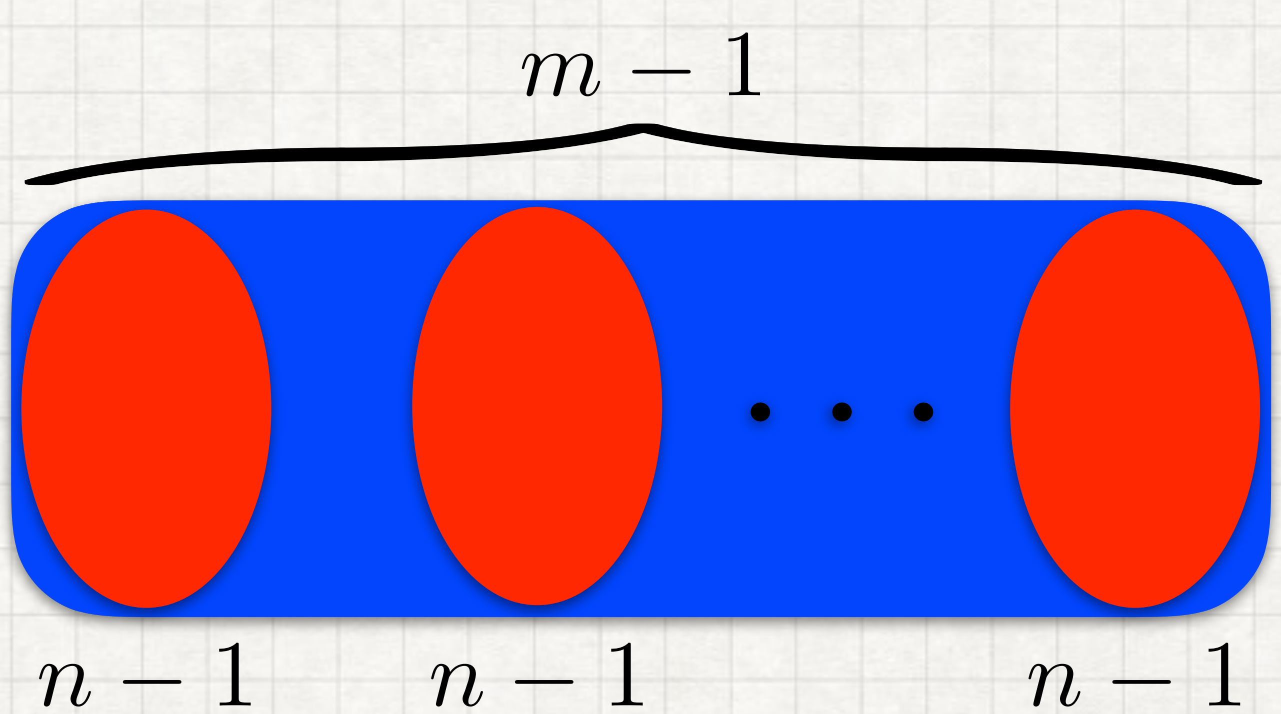
Lower bound construction:



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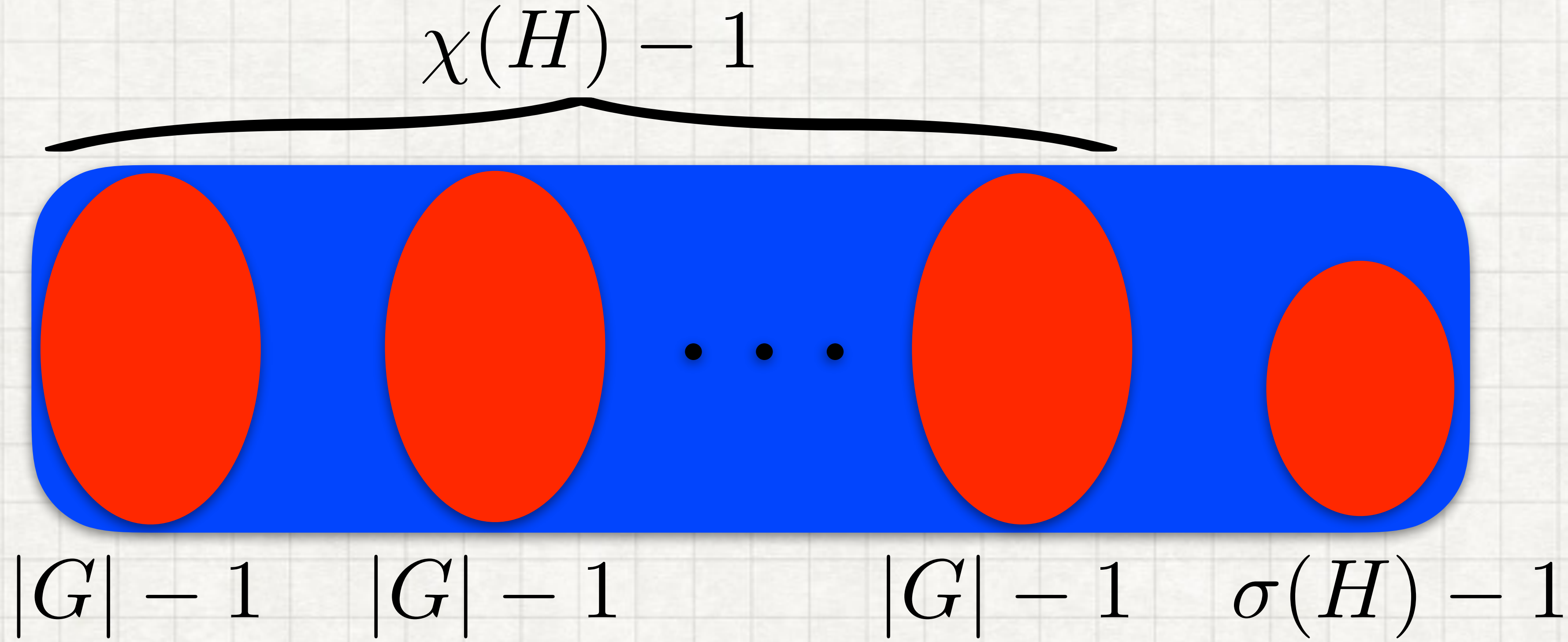

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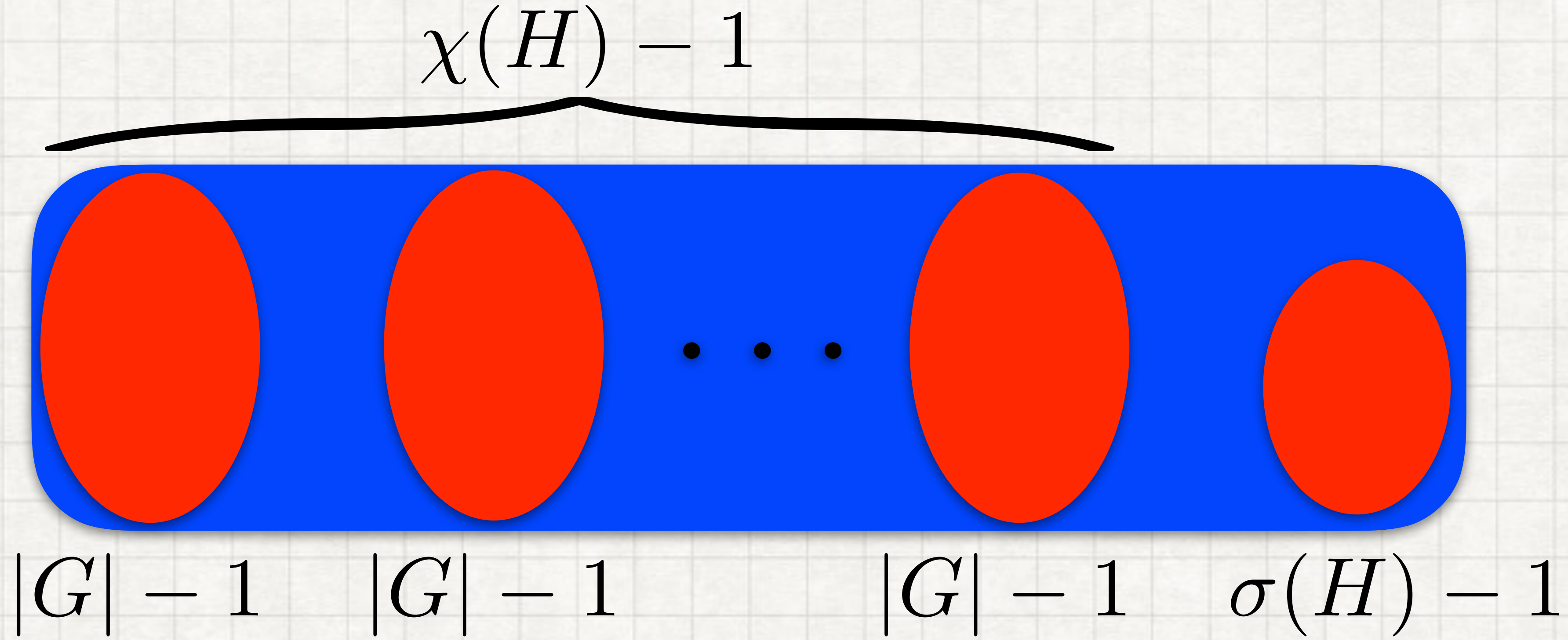

Definition: A graph G is called H-good if equality holds above.

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Theorem (B., Pokrovskiy, Sudakov 2016): The above theorem holds for $n_0 = \Omega(|H| \log^4 |H|)$.

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Theorem (Haxell 2001): In an expander on n vertices, we can find any bounded degree tree on $.99n$ vertices.

Theorem (Montgomery 2014): For any tree T on n vertices with max degree Δ, the random graph $G(n, \Delta \log^5 n/n)$ almost surely contains a copy of T.

Conjecture: For any tree T on n vertices with max degree Δ, and any graph H with $n \geq O(\Delta|H|)$,

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Thank you.