RAMSEY GOODNESS OF BOUNDED DEGREE TREES

Igor Balla

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Joint work with: Alexey Pokrovskiy, Benny Sudakov

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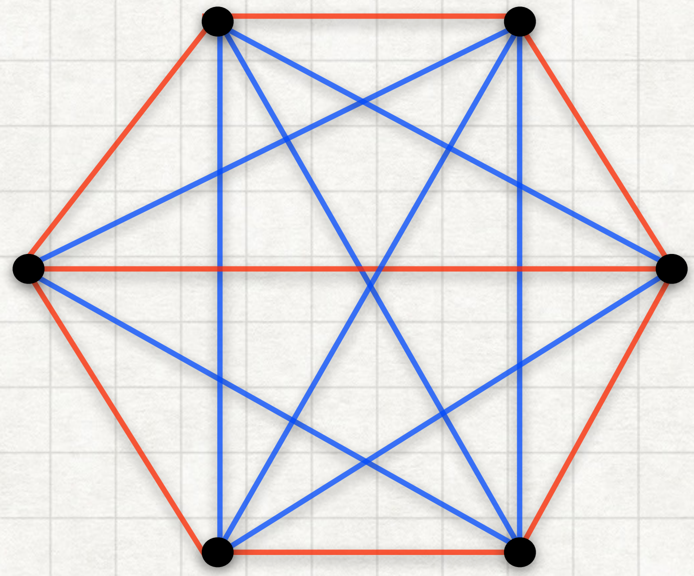

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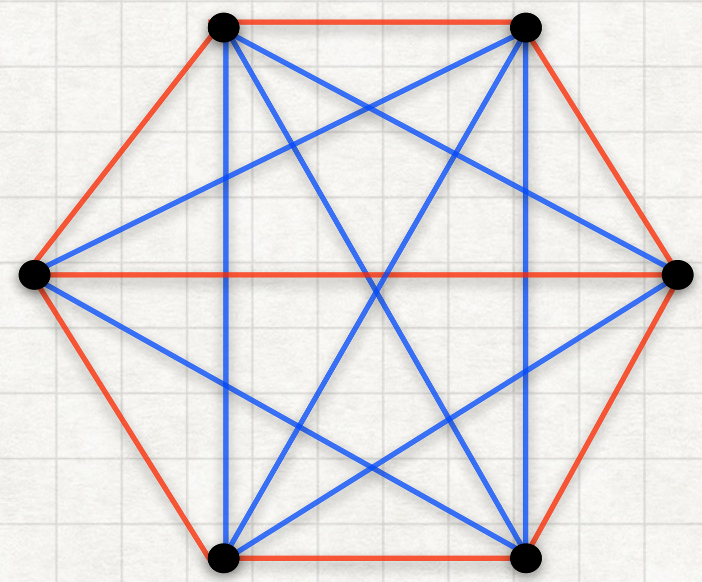

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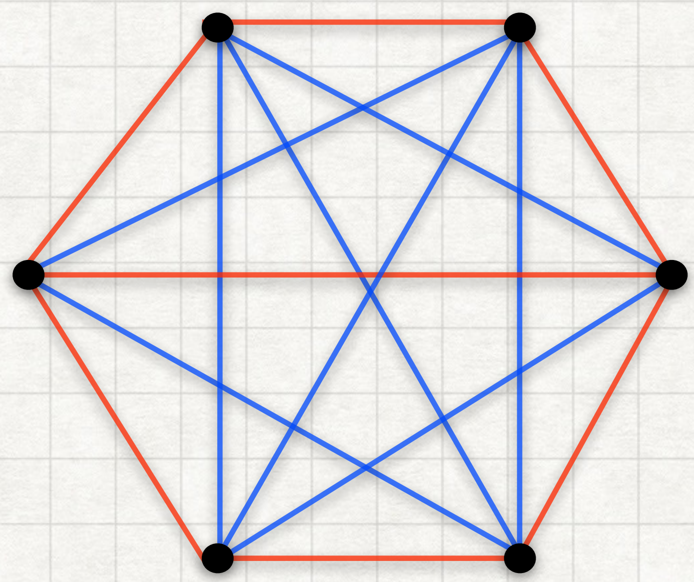

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Theorem (Erdős 1947; Erdős and Szekeres 1935):

$$\left(\sqrt{2}\right)^n \leq R(K_n, K_n) \leq 4^n$$

Theorem (Erdős 1947): $R(P_n, K_m) = (n-1)(m-1)+1$

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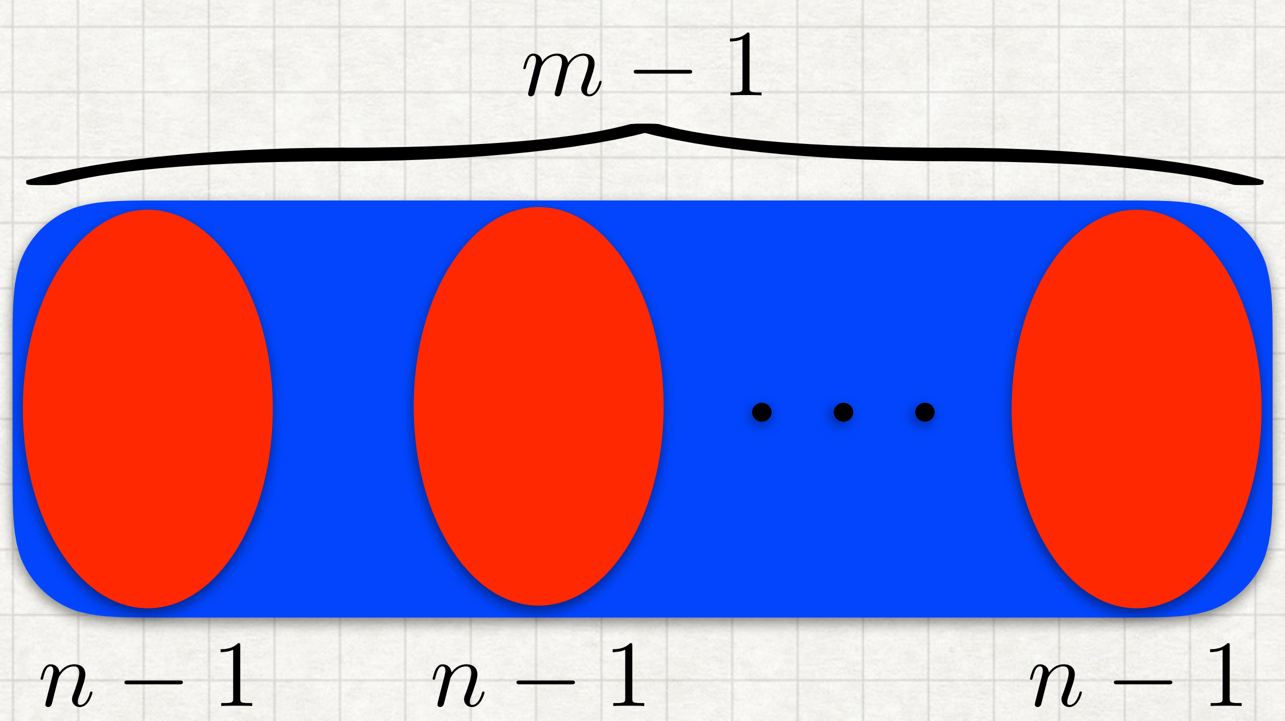
Lower bound construction:



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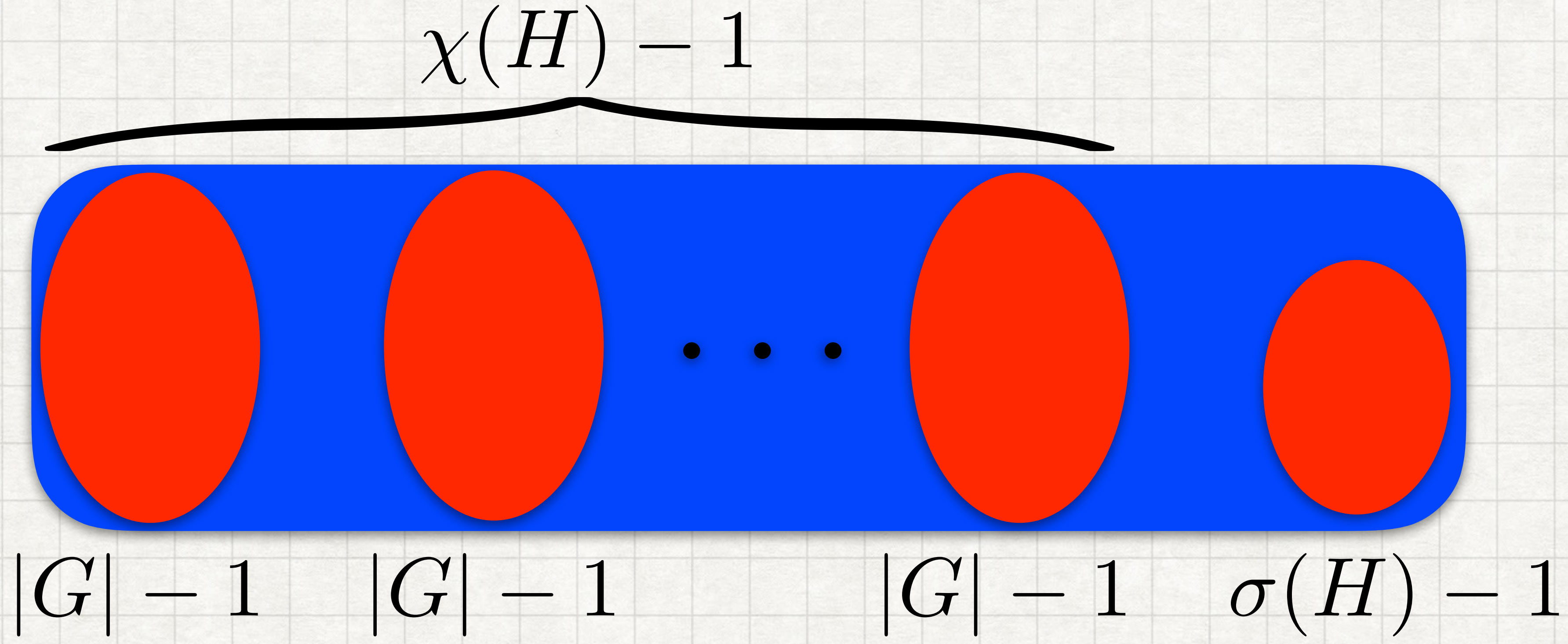

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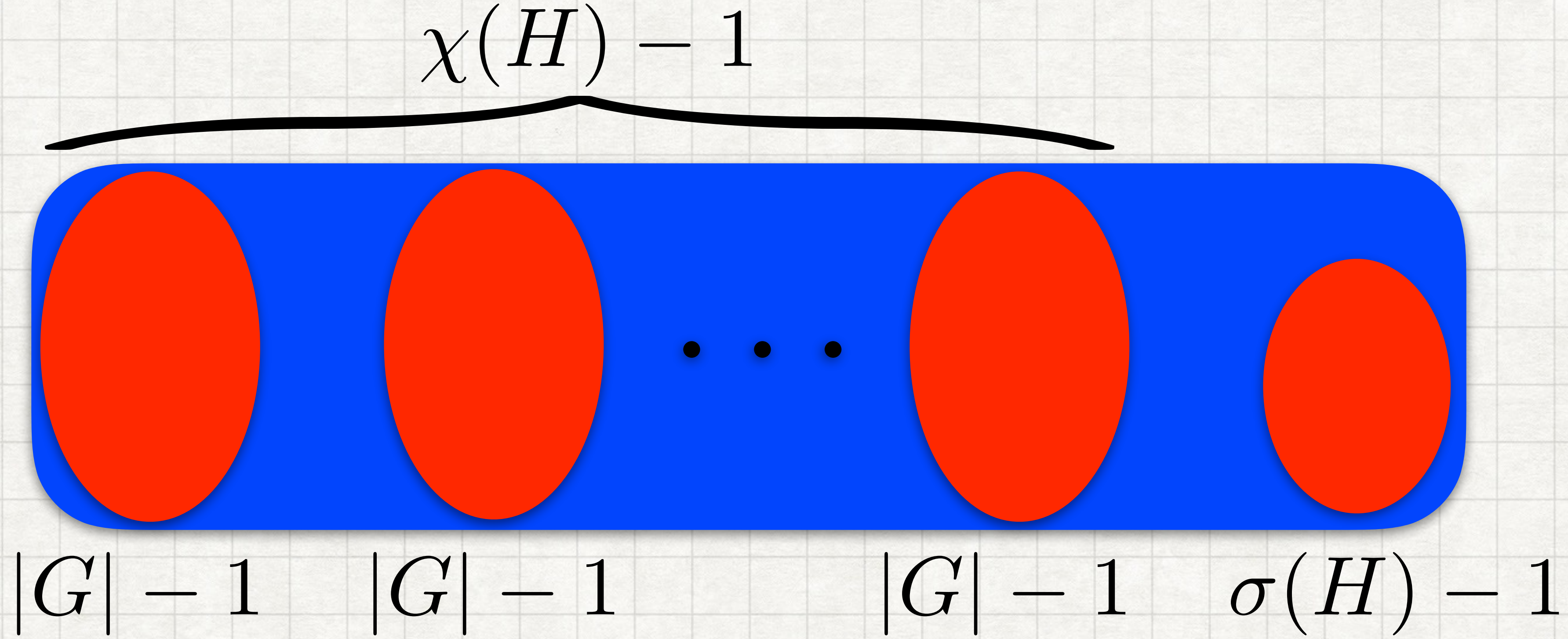

Definition: A graph G is called H-good if equality holds above.

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Theorem (B., Pokrovskiy, Sudakov 2016)**:** The above theorem holds for $n_0 = \Omega(|H| \log^4 |H|)$.

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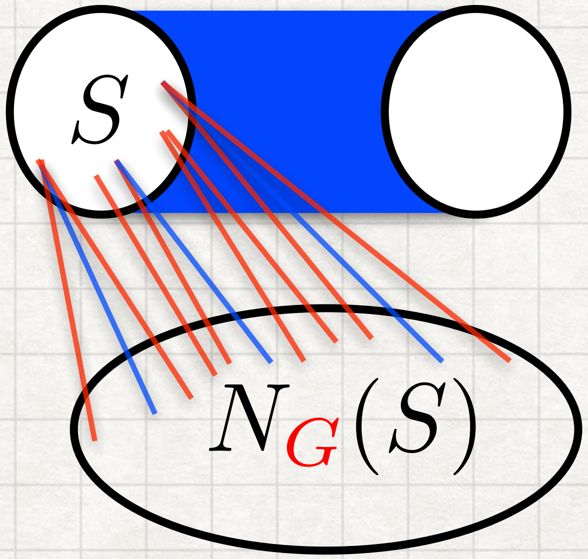

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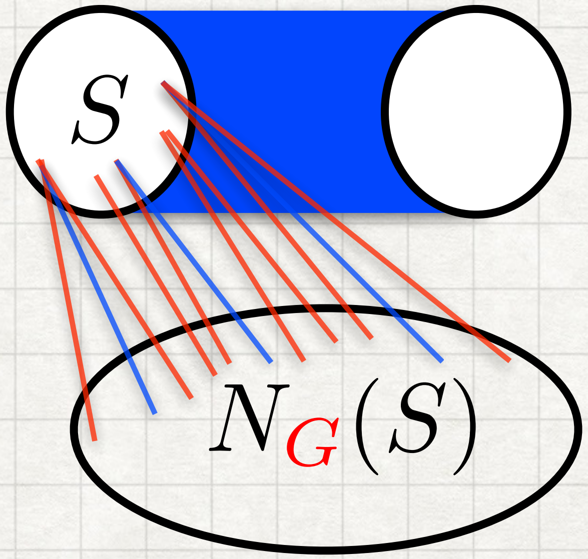

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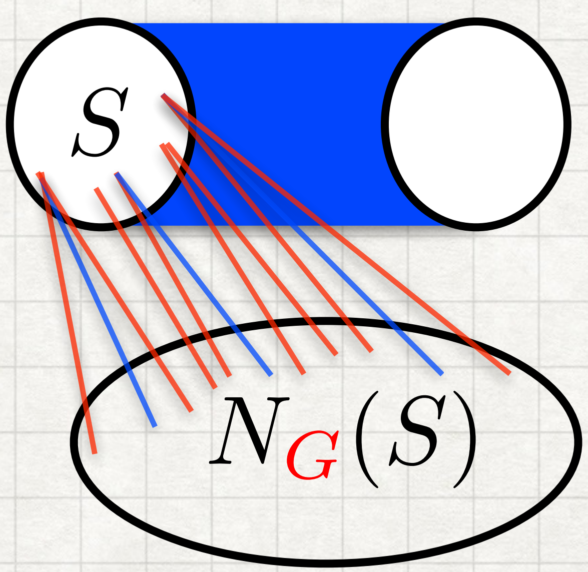

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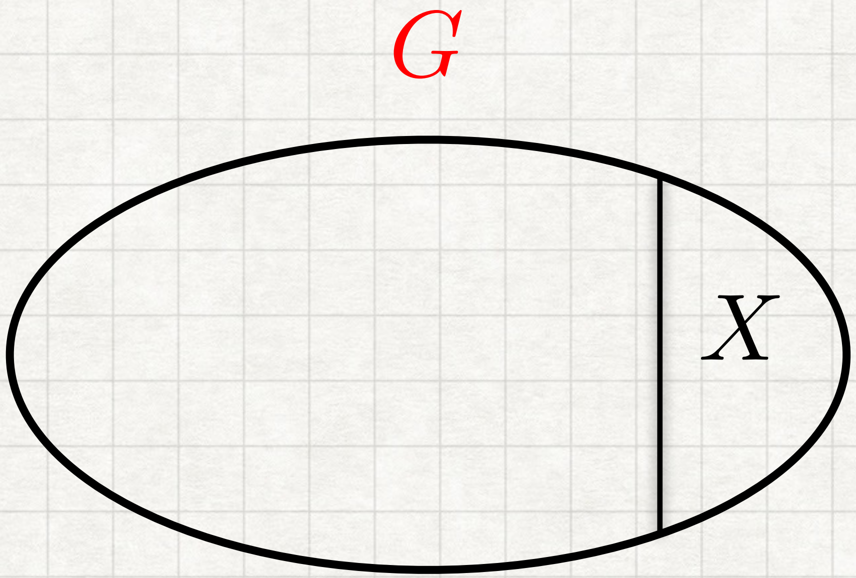

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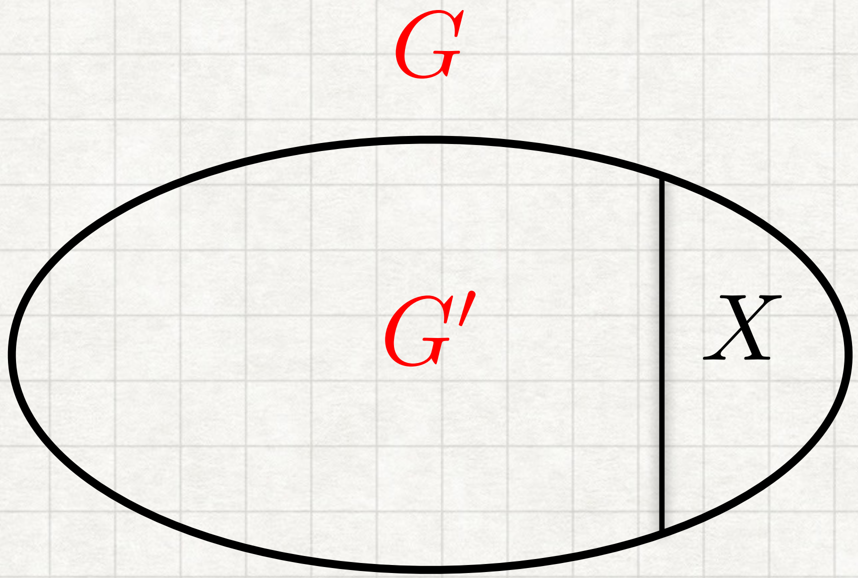

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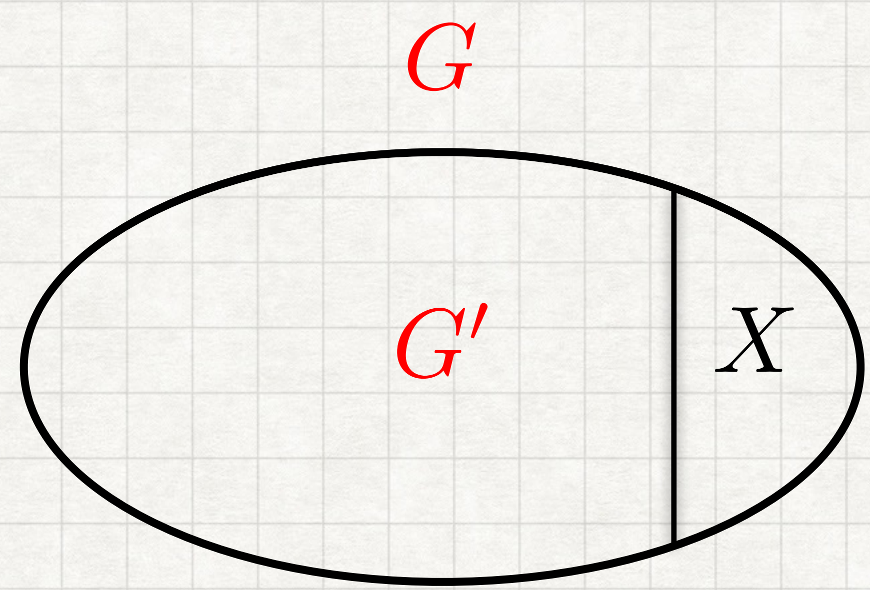

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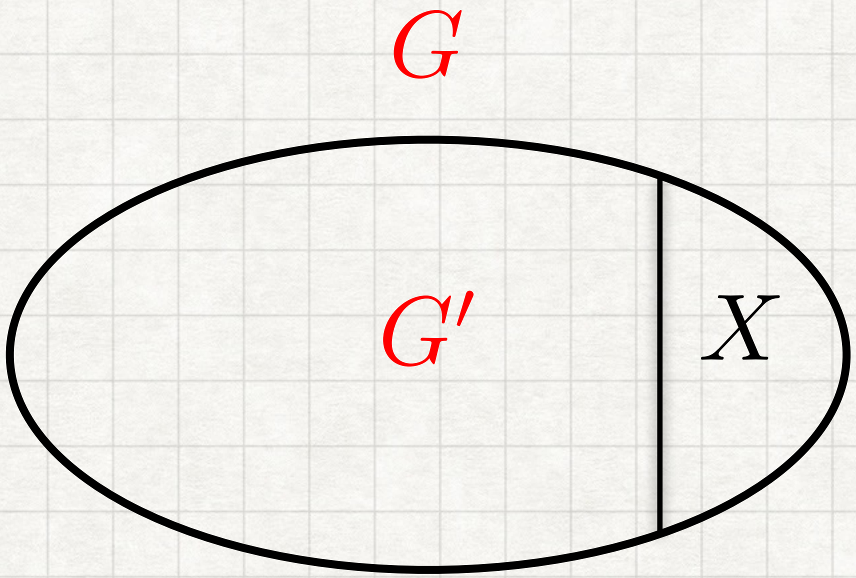

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(this is where the pesky $\log^4$ comes from)

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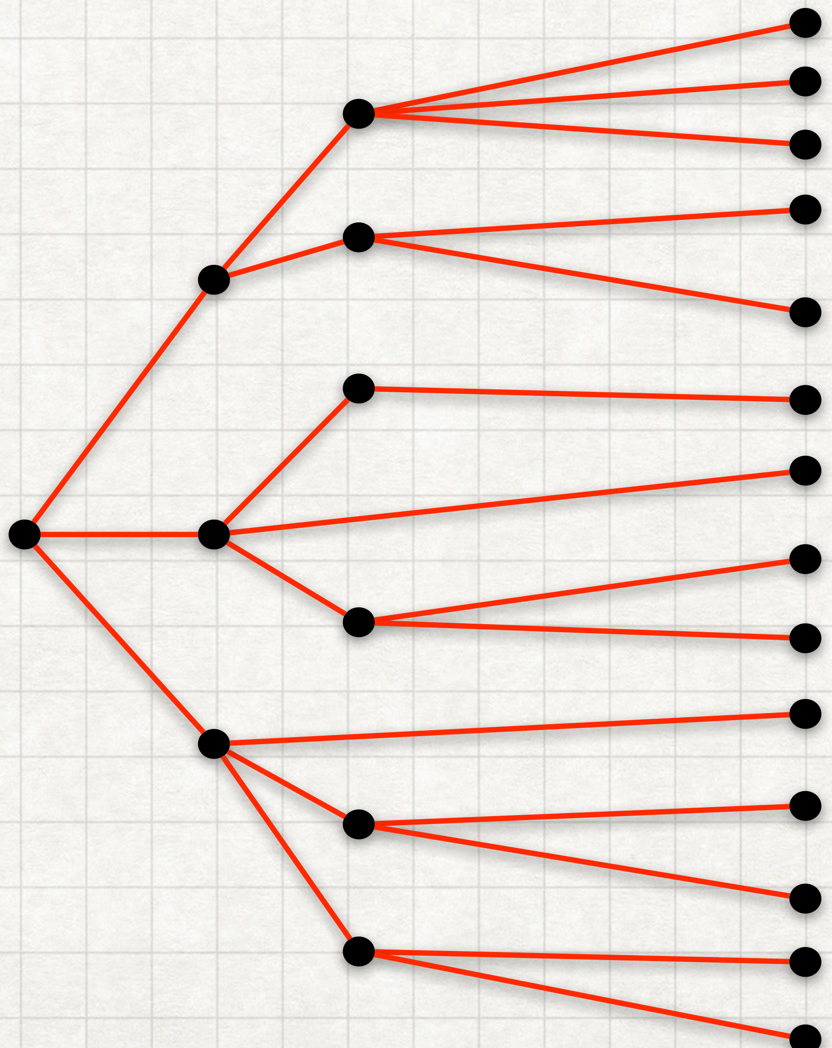

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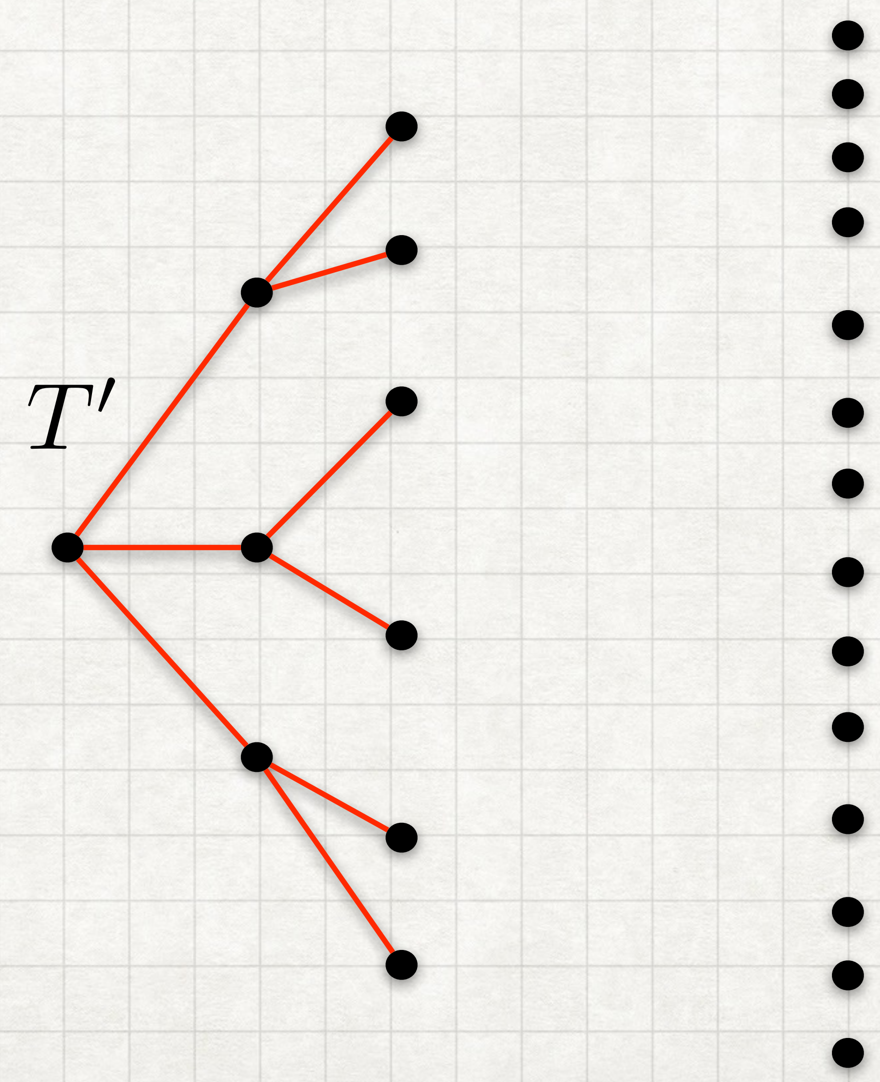

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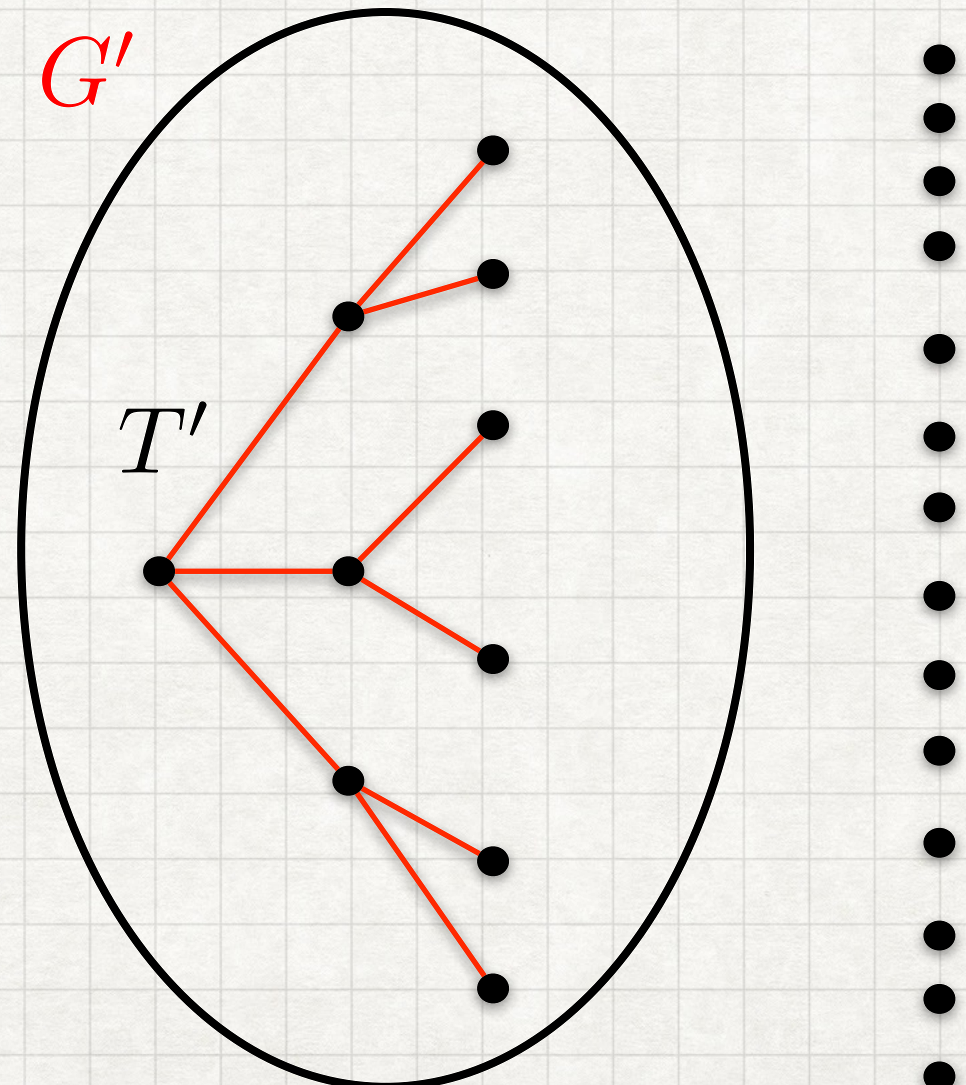

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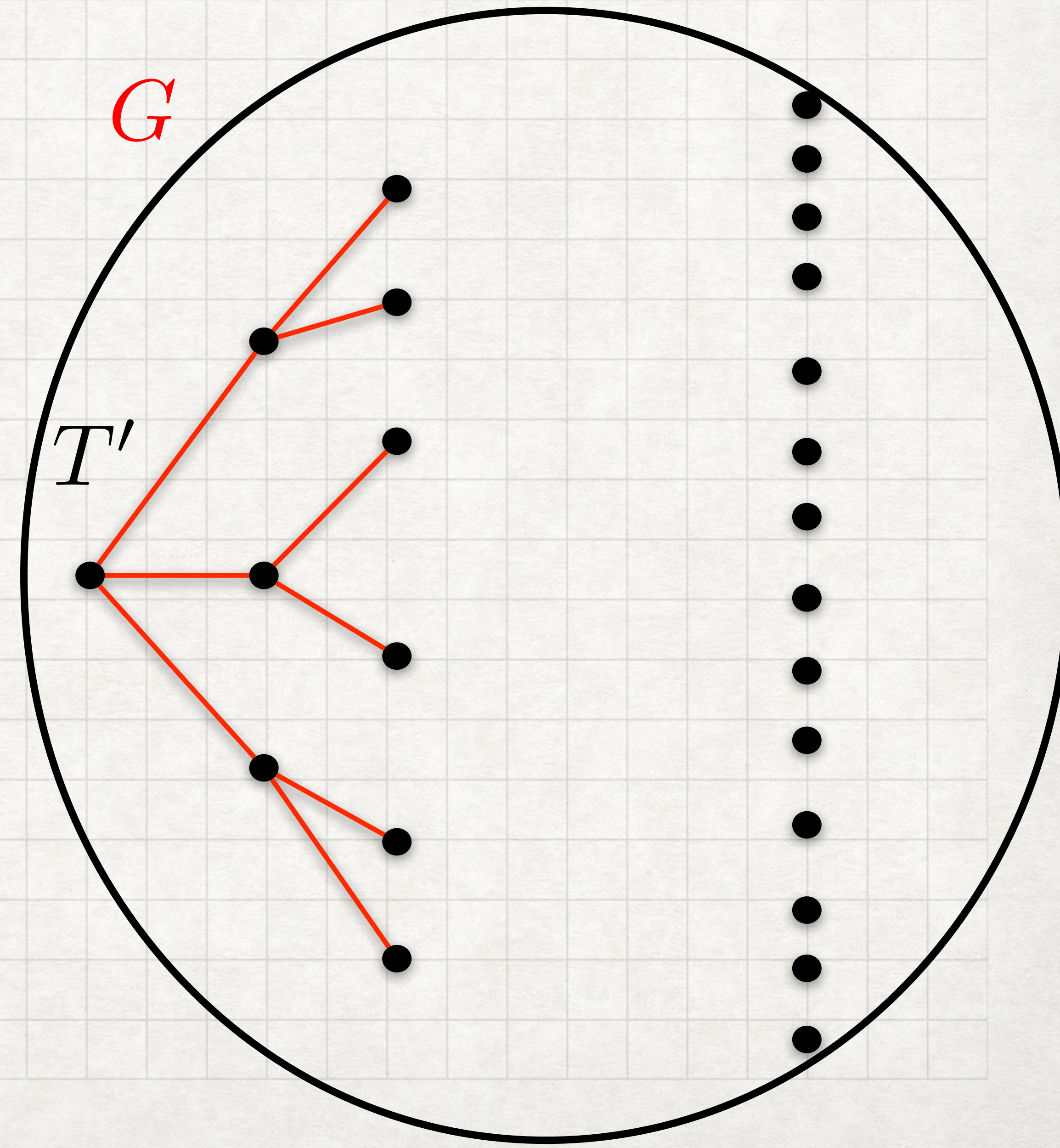

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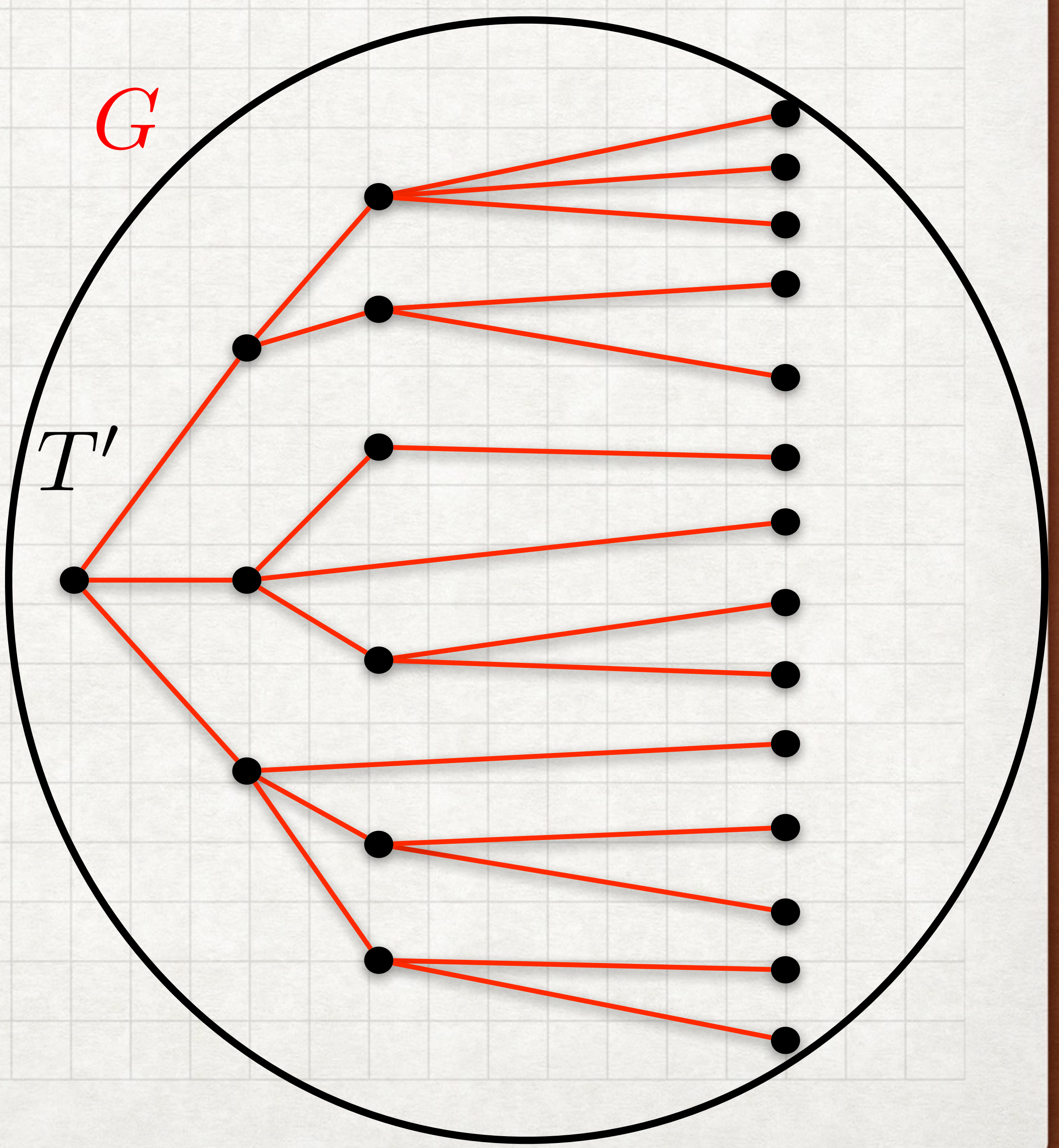
4. Apply Hall's theorem to connect the leaves and obtain T.



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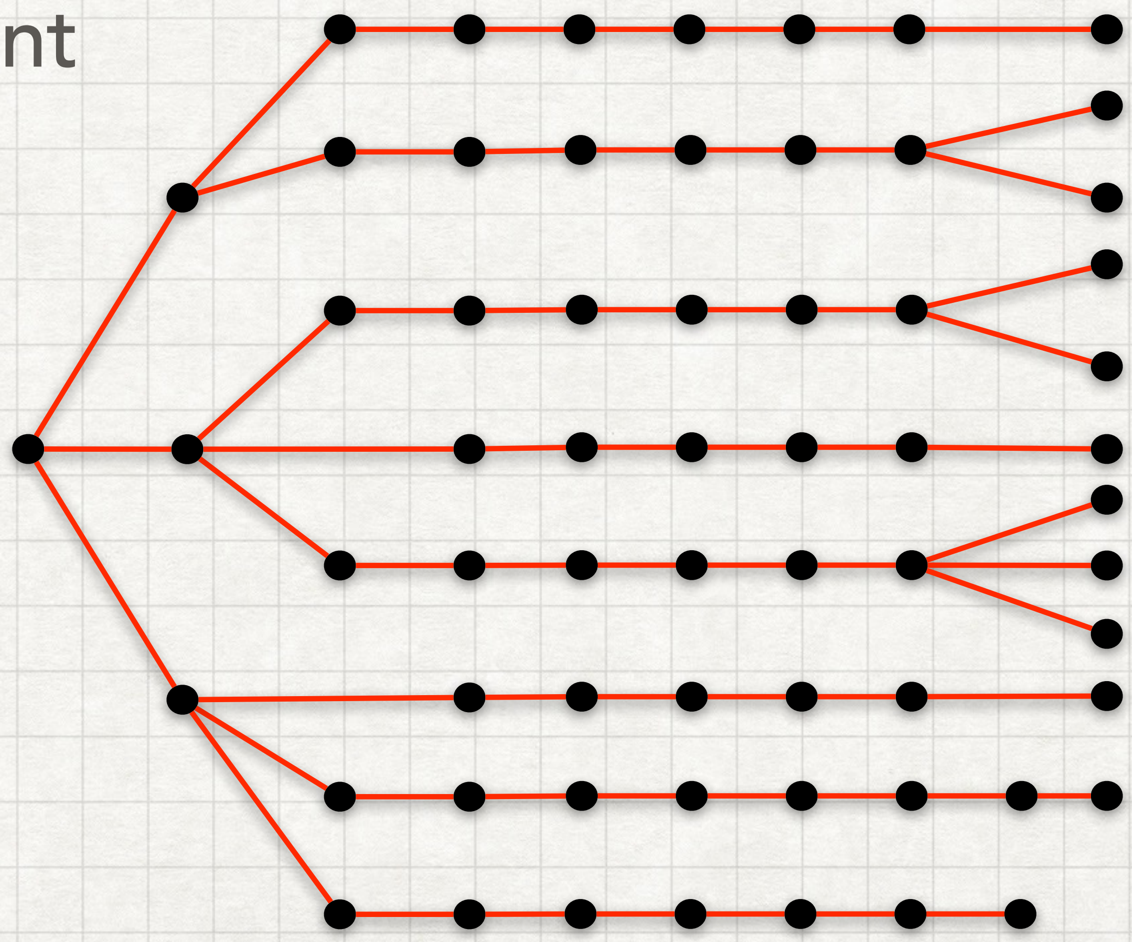

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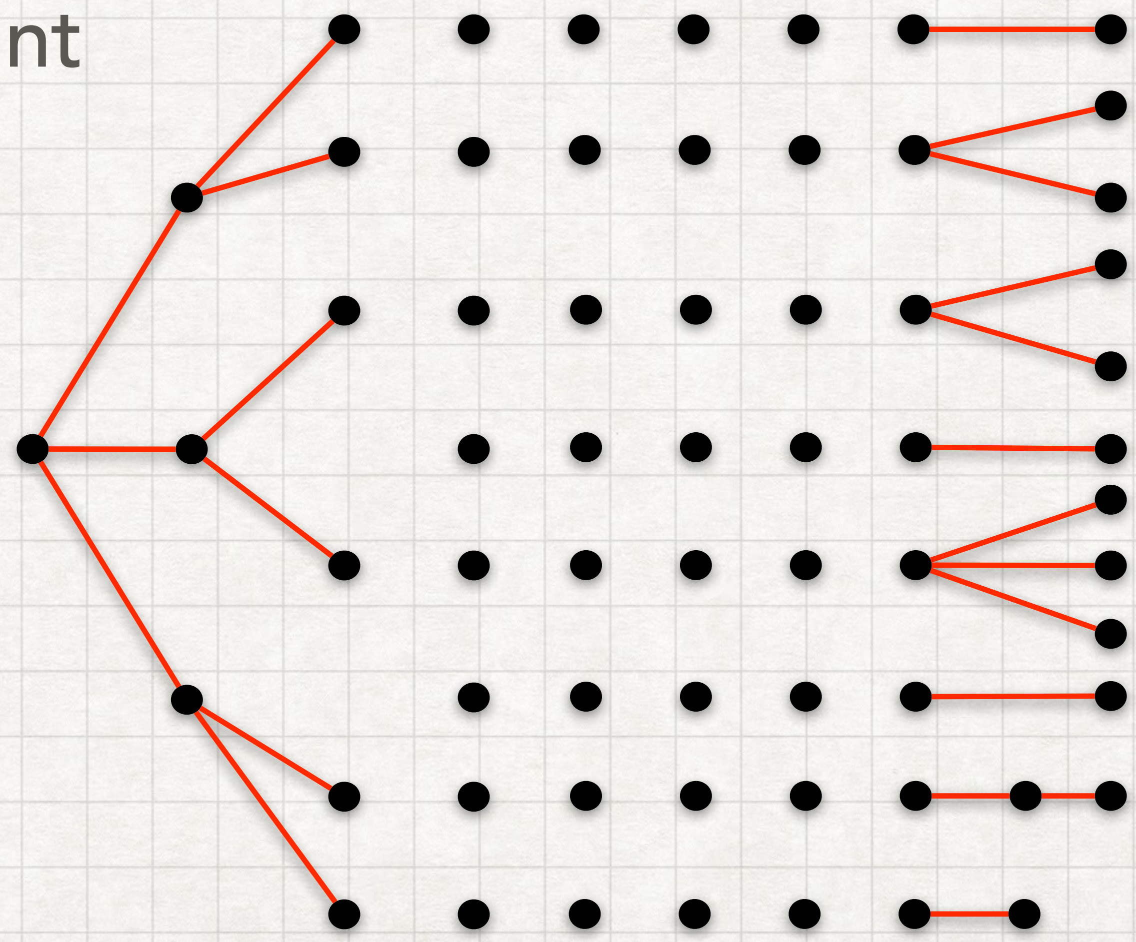

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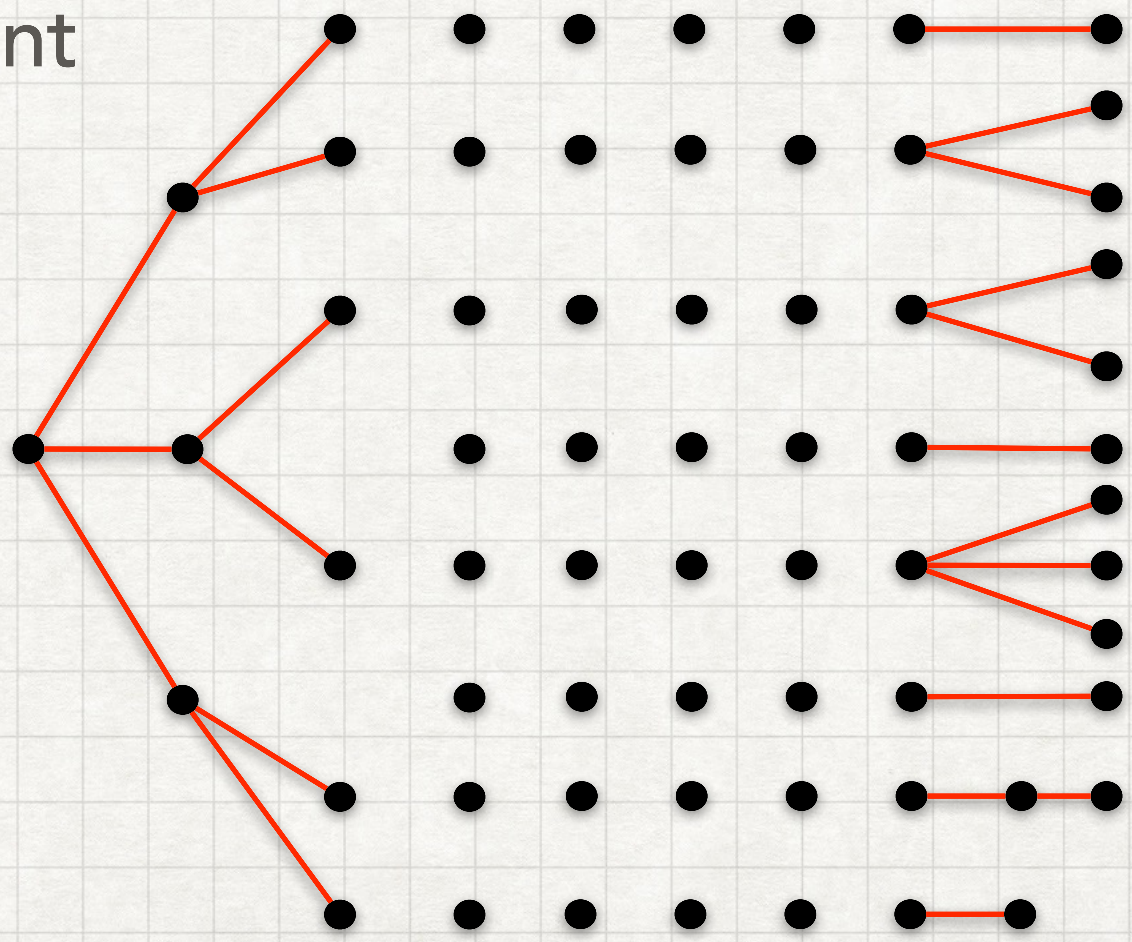

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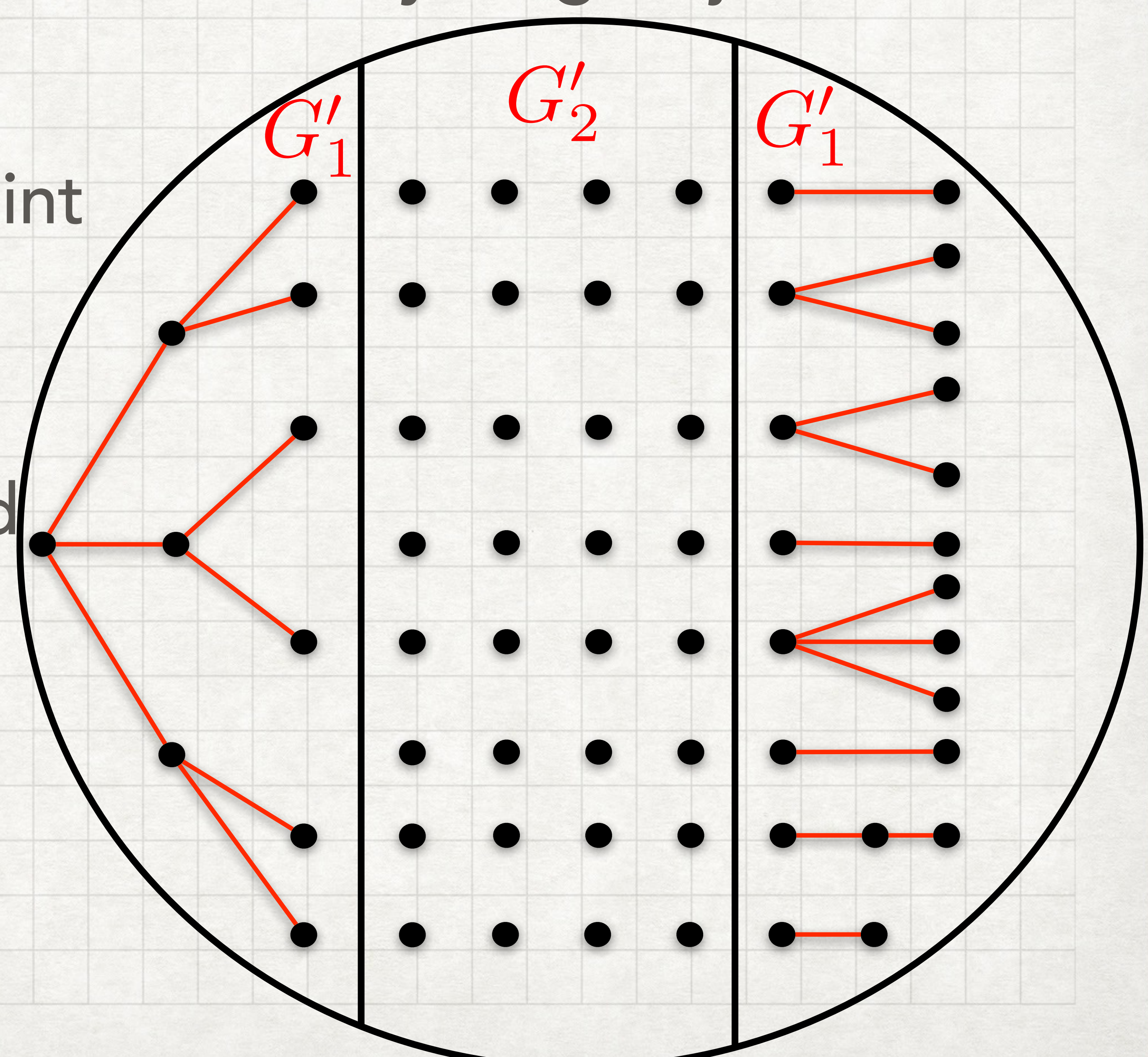

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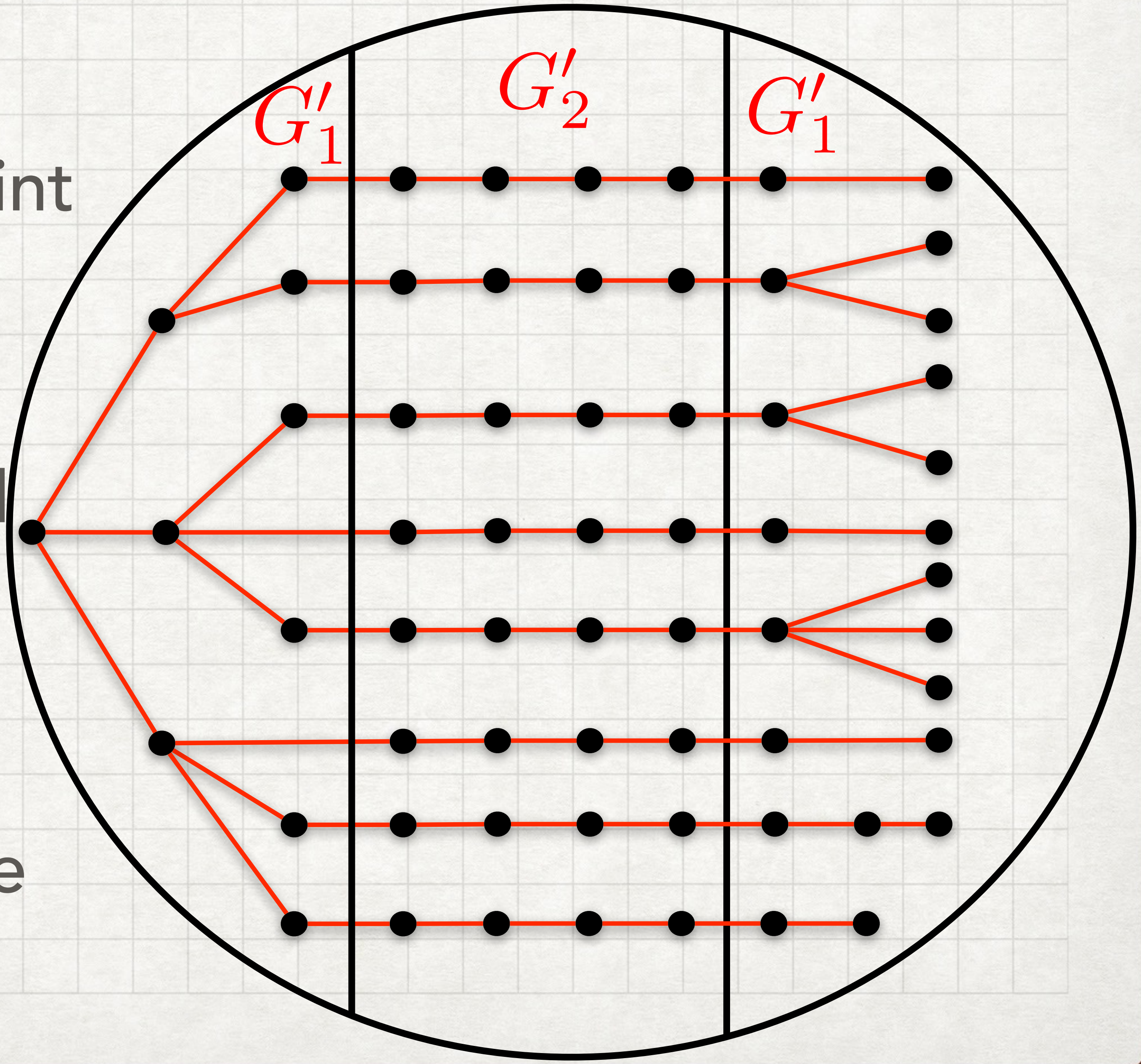

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- Our method proves it for trees with at least $\Omega(\Delta|H|)$ leaves.