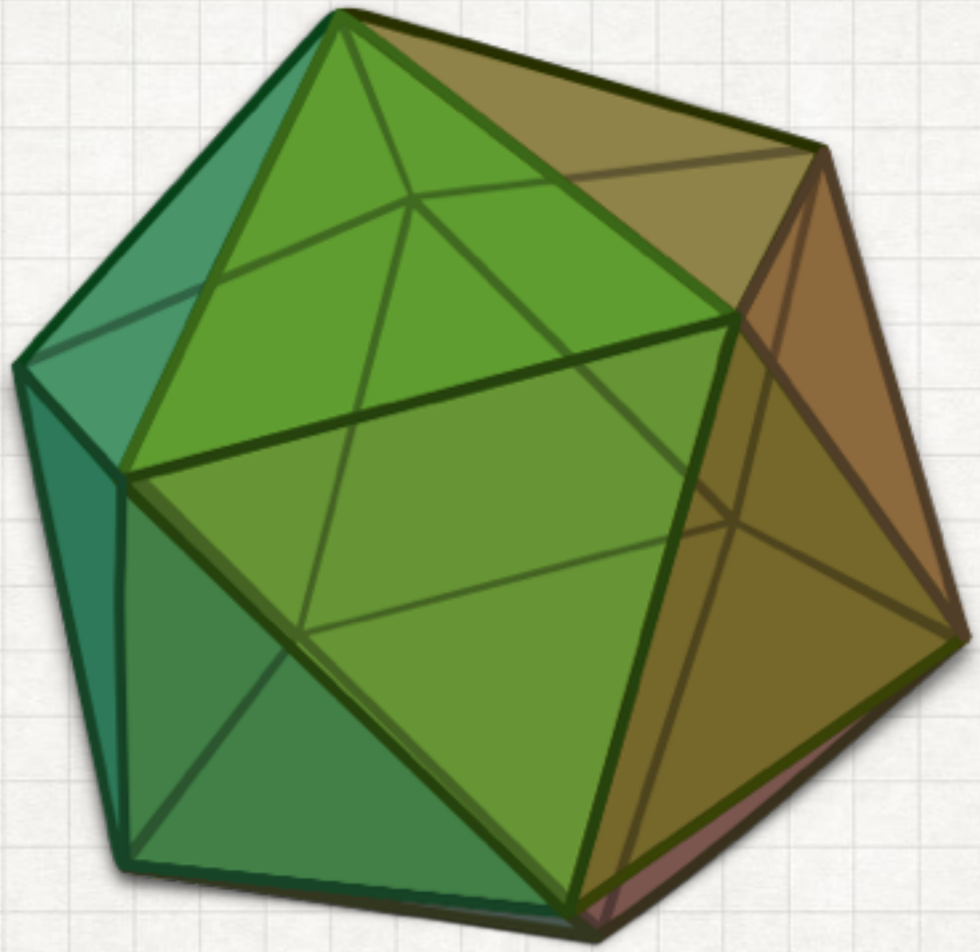
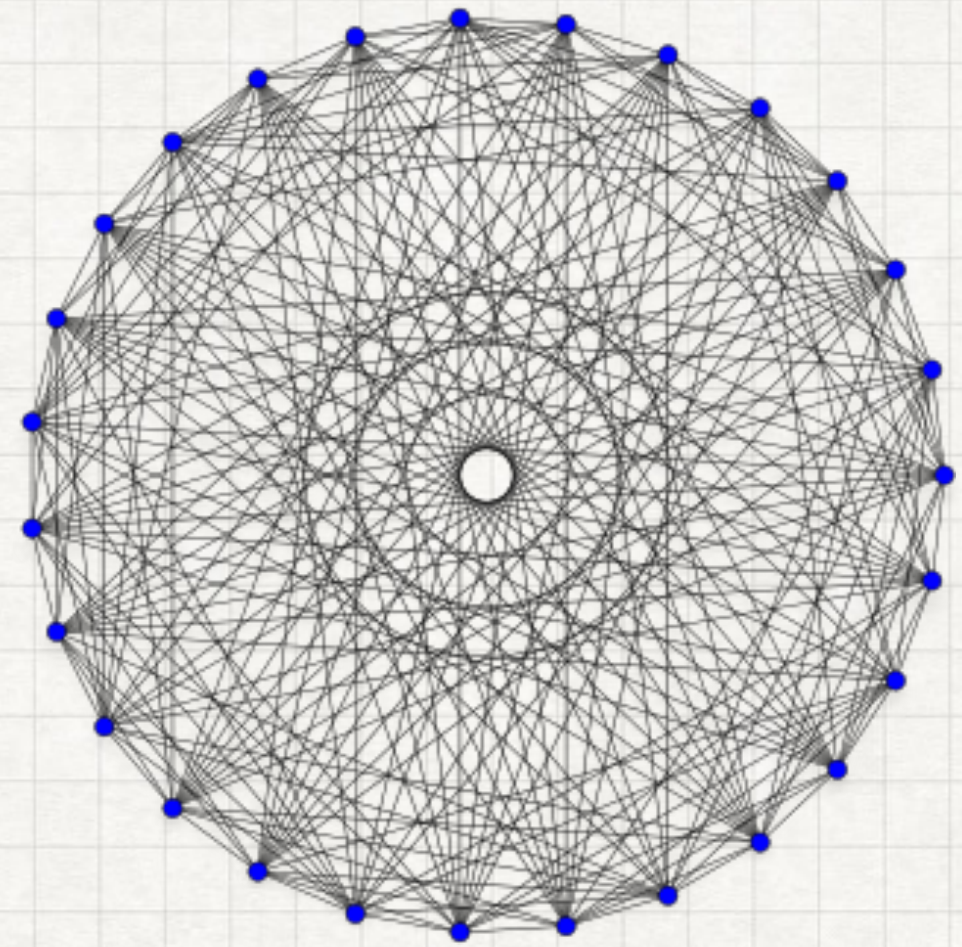




EQUIANGULAR LINES AND REGULAR GRAPHS

By: Igor Balla

Definition: A set of lines passing through the origin is called **equiangular**, if every pair of lines make the same angle.

Question: What is the maximum number of equiangular lines in $\mathbb{R}^r$?

Considered to be one of the founding problems of algebraic graph theory.

Connections:

- Elliptic geometry
- Frame theory
- Theory of polytopes
- Algebraic number theory
- Quantum information theory

Earliest work:

Haantjes, Seidel 47-48

Blumenthal 49

Van Lint, Seidel 66

Lemmens, Seidel 73

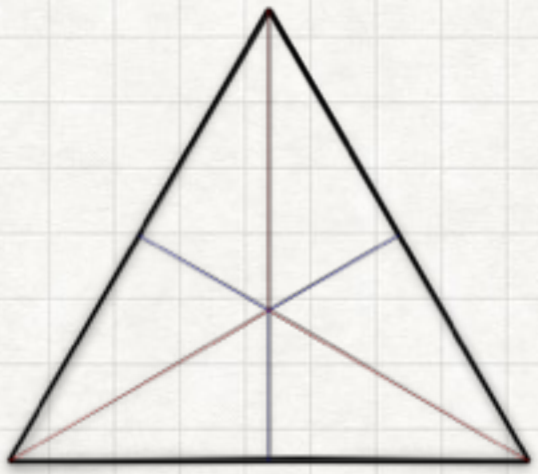
…

Examples

r = 2:

3 lines

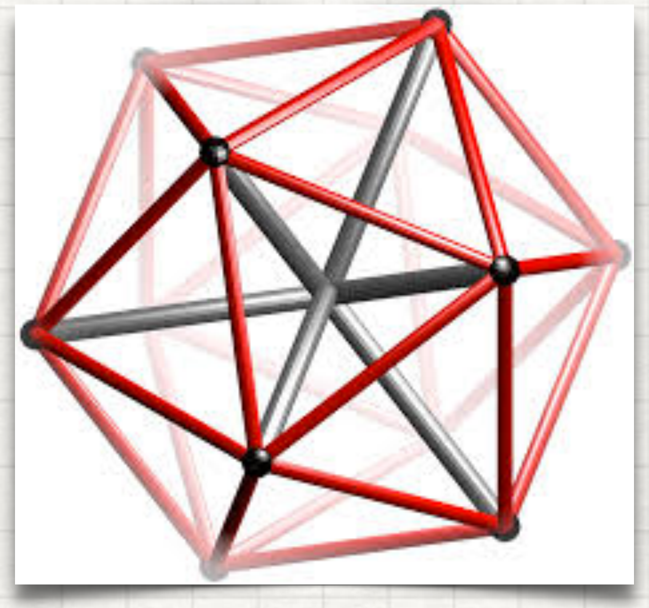
Triangle



r = 3:

6 lines

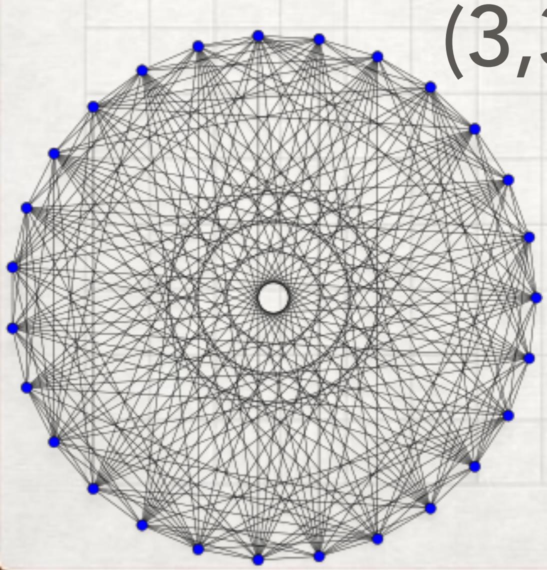
Icosahedron



r = 7:

28 lines

Take all 28 permutations of the vector (3,3,-1,-1,-1,-1,-1,-1).

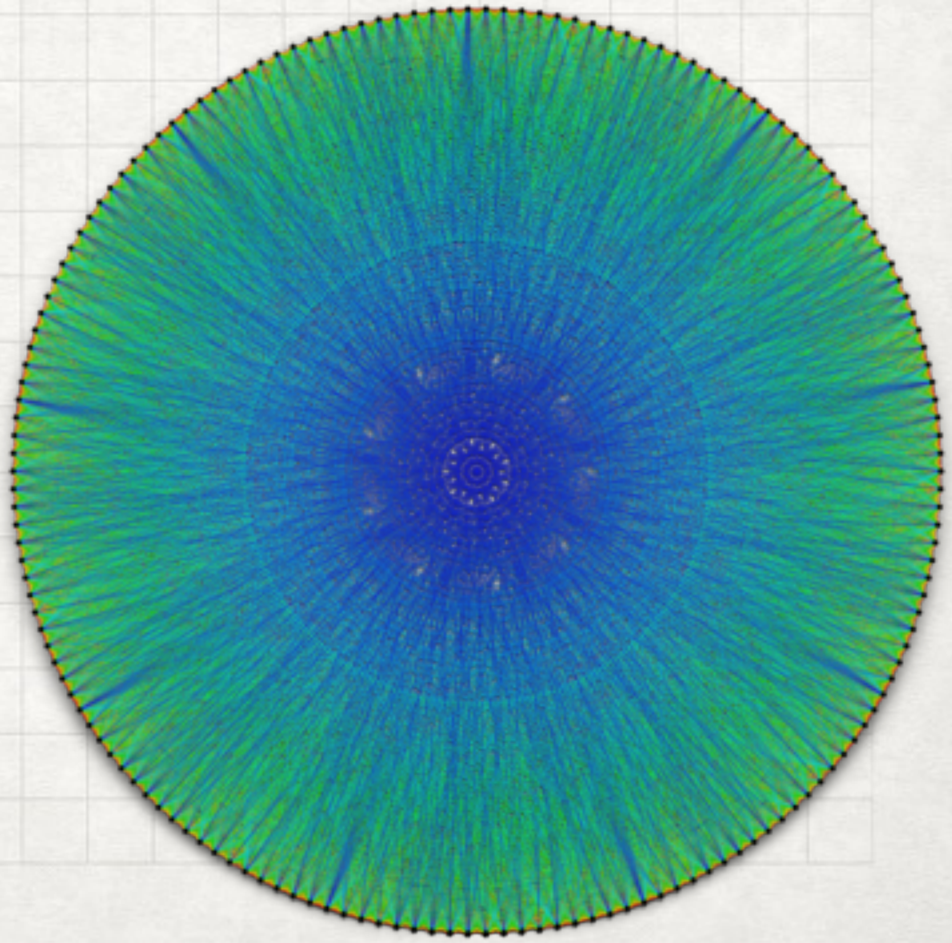


Schläfli Graph

r = 23:

276 lines

McLaughlin Graph



Theorem[Absolute bound] (Gerzon 73): The number of equiangular lines in $\mathbb{R}^r$ is at most $\binom{r+1}{2}$.

Proof: Let $v_1, \dots, v_n$ be unit vectors along the given lines. Then $\langle v_i, v_j \rangle = \pm\alpha$ for some $0 \leq \alpha < 1$.

Consider the matrices $v_1 v_1^\intercal, \dots, v_n v_n^\intercal$. They live in the $\binom{r+1}{2}$-dimensional space of symmetric matrices $\mathscr{S}_r$.

Recalling the Frobenius inner product of matrices

$$\langle A, B \rangle_F = \operatorname{tr}(A^\intercal B) = \sum_{i,j} A_{i,j} B_{i,j}$$

we have $\left\langle v_i v_i^\intercal, v_j v_j^\intercal \right\rangle_F = \operatorname{tr}(v_i v_i^\intercal v_j v_j^\intercal) = (v_i^\intercal v_j)^2 = \begin{cases} 1 & i = j \\ \alpha^2 & i \neq j \end{cases}$

Hence they are linearly independent. $\square$

Theorem (Gerzon 73): The number of equiangular lines in $\mathbb{R}^r$ is at most $\binom{r+1}{2}$.

This bound is tight in dimension 2,3, 7 and also 23. No other cases are known.

Theorem (de Caen '00 / Jedwab, Wiebe '15 / Greaves, et al. '15): There exist $\Omega(r^2)$ equiangular lines in $\mathbb{R}^r$.

These constructions all have $\alpha = \Theta\left(\frac{1}{\sqrt{r}}\right)$.

Theorem (Neumann 73): If $n > 2r$ then $\frac{1-\alpha}{2\alpha}$ is an integer.

Question (Lemmens, Seidel 73):
Determine $N_\alpha(r)$, the maximum number of equiangular lines in $\mathbb{R}^r$ with common angle $\arccos(\alpha)$?

What is known?

Theorem[Relative Bound] (Lemmens, Seidel 73): $N_\alpha(r) \le r\frac{1-\alpha^2}{1-r\alpha^2}$ for all $\alpha < \frac{1}{\sqrt{r}}$.

Theorem (B., Dräxler, Keevash, Sudakov 17): $N_\alpha(r) \le 2r-2$ for all $\alpha \gg \frac{1}{\sqrt{\log r}}$ with equality if and only if $\alpha = 1/3$.

Question: What about for $\frac{1}{\sqrt{r}} \le \alpha \le O\left(\frac{1}{\sqrt{\log r}}\right)$?

Theorem (Glazyrin, Yu 18): $N_\alpha(r) \le \left(\frac{2}{3\alpha^2} + \frac{4}{7}\right) r + 2$ for all $\alpha \le \frac{1}{3}$.

Theorem (Jiang, Tidor, Yao, Zhang, Zhao 19): Let k be the minimum number of vertices in a graph with spectral radius $\frac{1-\alpha}{2\alpha}$. Then $N_\alpha(r) = \left\lfloor \frac{k(r-1)}{(k-1)} \right\rfloor$ for all $\alpha \ge \frac{Ck}{\log\log r}$, where $C > 0$ is a constant.

New results

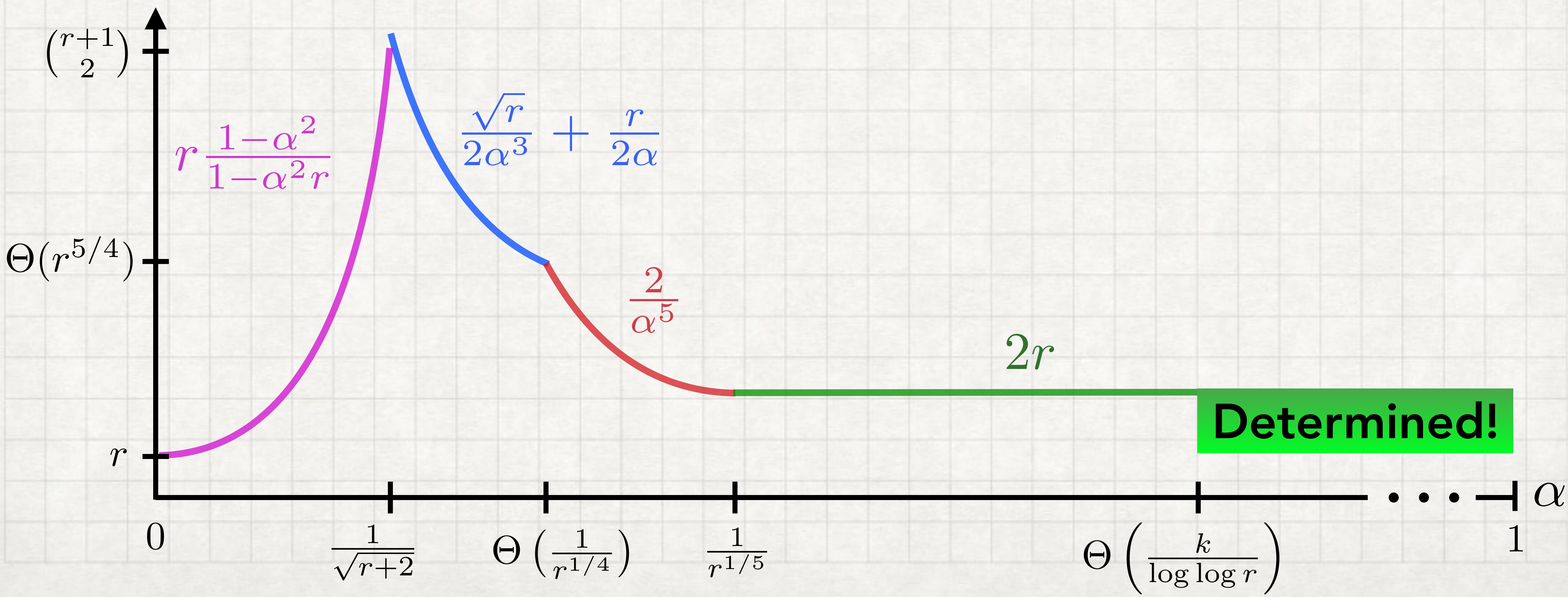
Theorem(B.): $N_\alpha(r) \le \frac{\sqrt{r}}{2\alpha^3} + \frac{(1+\alpha)r}{2\alpha}$.

- Based on bounding eigenvalues of a Gram matrix.

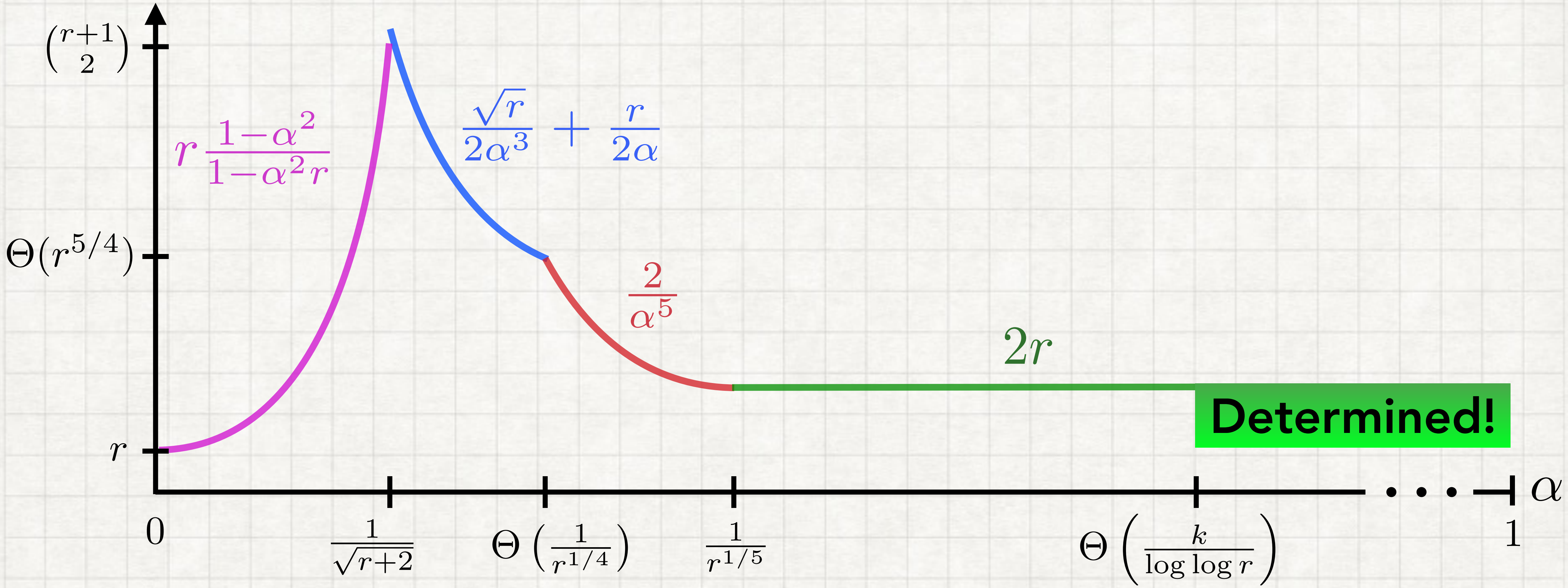
Theorem(B.): $N_\alpha(r) \le \max\left(\frac{2}{\alpha^5} + \frac{2}{\alpha^3(1-\alpha)}, \left(2 + \frac{8\alpha^2}{(1-\alpha)^2}\right)(r+1)\right)$

$$= (1+o(1))\max\left(\tfrac{2}{\alpha^5}, 2r\right).$$

- Based on bounding degrees of a graph, bootstrapped via an Alon-Boppana theorem.



More new results



Theorem(B.): Let $s \geq 2$ be an integer and suppose that $\alpha \gg \frac{1}{r^{1/(2s+1)}}$. Then $N_\alpha(r) \leq (1+o(1))\left(1+\frac{1}{4\cos^2\left(\frac{\pi}{s+2}\right)}\right)r$.

In particular, if $s \to \infty$ then $N_\alpha(r) \leq (1+o(1))\frac{5}{4}r$.

- Based on improved Alon-Boppana theorem (Jiang-Polyanskii)

New results in the complex setting

Given a pair of complex lines $U, V \subset \mathbb{C}^r$, the quantity $|\langle u, v\rangle|$ is the same for all unit vectors $u \in U, v \in V$ and so $\arccos|\langle u, v\rangle|$ is called the **Hermitian angle** between U and V.

We define $N_\alpha^{\mathbb{C}}(r)$ to be the maximum number of complex equiangular lines in $\mathbb{C}^r$ with common Hermitian angle $\arccos(\alpha)$.

Theorem[Absolute bound] (Delsarte, Goethals, Seidel 75): $N_\alpha^{\mathbb{C}}(r) \le r^2$

Collections of r^2 complex equiangular lines in $\mathbb{C}^r$ are known as SICs/SIC-POVMs and their existence has great theoretical and practical importance in quantum theory.

New results in the complex setting

Conjecture (Zauner 99): For each $r \in \mathbb{N}$, $\max_\alpha N_\alpha^{\mathbb{C}}(r) = r^2$ and a construction can be obtained as the orbit of a vector under the action of a Weyl-Heisenberg group.

Theorem[Relative Bound] (Delsarte, Goethals, Seidel 75): $N_\alpha^{\mathbb{C}}(r) \leq r\frac{1-\alpha^2}{1-r\alpha^2}$ for all $\alpha < \frac{1}{\sqrt{r}}$.

Theorem(B.): $N_\alpha^{\mathbb{C}}(r) \leq \frac{\sqrt{r}}{\alpha^3} + \frac{(1+\alpha)r}{\alpha}$.

Theorem[Improved Welch](B.): Given unit vectors $v_1, \ldots, v_n \in \mathbb{C}^r$ let H be the $n \times n$ matrix defined by $H_{i,j} = |\langle v_i, v_j \rangle|^2$. Then $\mathbb{1}^\intercal H^\dagger \mathbb{1} \leq r$, where $H^\dagger$ is the Moore-Penrose generalised inverse, and moreover $\sum_{i,j} |\langle v_i, v_j \rangle|^2 \geq \frac{n^2}{r}\left(2 - \frac{\mathbb{1}^\intercal H^\dagger \mathbb{1}}{r}\right)$.

New results for regular graphs

Theorem(B.): Let G be a k-regular graph with second and last eigenvalue λ_2, λ_n. If the spectral gap satisfies $k - \lambda_2 < \frac{n}{2}$, then

$$\boldsymbol{k} < 2\left(k - \frac{(k-\lambda_2)^2}{n}\right) \leq \frac{\lambda_2(\lambda_2+1)(2\lambda_2+1)}{1-\frac{2(k-\lambda_2)}{n}} - \lambda_2(3\lambda_2+1),$$

$$-\lambda_n \leq \frac{\lambda_2(\lambda_2+1)}{1-\frac{2(k-\lambda_2)}{n}} - \lambda_2,$$

with equality in both whenever $n + 1 = \binom{n-\text{mult}(\lambda_2)+1}{2}$, i.e. when G corresponds to a set of equiangular lines meeting the absolute bound.

Corollary(B.): If $k - \lambda_2 \ll n$, then $k \leq (1+o(1))\lambda_2^3$ and $-\lambda_n \leq (1+o(1))\lambda_2^2$.

In particular, if G is bipartite we have $\lambda_2 \geq (1-o(1))\sqrt{k}$.

Theorem[Relative Bound] (Lemmens, Seidel 73): $N_\alpha(r) \le r\frac{1-\alpha^2}{1-r\alpha^2}$ for all $\alpha < \frac{1}{\sqrt{r}}$.

Proof: Using the Frobenius inner product, orthogonally project the $r \times r$ identity matrix I onto the span of $v_1 v_1^\intercal, \ldots, v_n v_n^\intercal$.

Its squared Frobenius norm changes from $\mathrm{tr}(I^\intercal I) = r$ to $\frac{n}{\alpha^2 n + 1 - \alpha^2}$. $\square$

A Really New Idea: Let V be the $r \times n$ matrix with ith column v_i, so that $Vx = x_1 v_1 + \ldots, x_n v_n$.

Project **symmetric** matrices of the form $Vx(Vy)^\intercal + Vy(Vx)^\intercal$.

Previous approaches in the case $\alpha \gg \frac{1}{\log r}$

Relied on applying Ramsey's theorem to find a large independent set in the graph G with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G) = \{v_i v_j : \langle v_i, v_j \rangle = -\alpha\}$.

An independent set of this graph corresponds to a set of vertices with all pairwise inner products $= \alpha$, i.e. a simplex.

Projecting vertices onto the orthogonal complement of the span of this simplex allowed one to bound Δ, the max degree of the graph G.

New Idea: Instead of looking for such a simplex in the vertices, recall that $v_1 v_1^\intercal, \ldots, v_n v_n^\intercal$ already form a large simplex with respect to the Frobenius inner product.

Project onto it!

A new inequality

Definition: Given $f : \mathbb{R} \to \mathbb{R}$ and $v \in \mathbb{R}^n$ we define $f(v) \in \mathbb{R}^n$ to be the vector with ith coordinate $f(v_i)$.

Theorem(B.): Let $M = V^\intercal V$ be the Gram matrix of $v_1, \ldots, v_n$ and let $f(x) = x^2$. Then for all $x, y \in \mathbb{R}^n$

$$\frac{1-\alpha^2}{2}\left(\langle x, Mx\rangle \langle y, My\rangle + \langle x, My\rangle^2\right) + \frac{\alpha^2}{\alpha^2 n + 1 - \alpha^2}\langle Mx, My\rangle^2 \geq \langle f(Mx), f(My)\rangle$$

with equality whenever $n = \binom{r+1}{2}$.

Taking x, y to be eigenvectors of M or standard basis vectors yields bounds on eigenvalues of M or the degrees of G.

(Finite-dimensional) Hilbert spaces

Recall that $\mathscr{S}_r$ is the Hilbert space of $r \times r$ symmetric matrices with respect to the Frobenius inner product.

Definition: Given a linear map $L : \mathbb{R}^n \to \mathscr{S}_r$, we let $L^\# : \mathscr{S}_r \to \mathbb{R}_n$ denote the *adjoint map* which is defined by $\langle L^\# M, v \rangle = \langle M, Lv \rangle_F$ for all $v \in \mathbb{R}^n, M \in \mathscr{S}_r$. For a single matrix $X \in \mathscr{S}_r$, we define $X^\# = L^\#$ where $L : \mathbb{R}^1 \to \mathscr{S}_r$ is given by $Le_1 = X$.

If we identify each $r \times r$ matrix with a vector in $\mathbb{R}^{r^2}$, then L becomes an $r^2 \times n$ matrix and $L^\#$ becomes its $n \times r^2$ transpose.

Fact: Let $W_1, \ldots, W_n \in \mathscr{S}_r$ and define $\mathscr{W} : \mathbb{R}^n \to \mathscr{S}_r$ by $\mathscr{W} e_i = W_i$. Then $\mathscr{W}(\mathscr{W}^\# \mathscr{W})^{-1} \mathscr{W}^\#$ is the orthogonal projection (with respect to the Frobenius inner product) onto the span of $W_1, \ldots, W_n$.

Theorem(B.): Let $M = V^\intercal V$ be the Gram matrix of $v_1, \ldots, v_n$ and let $f(x) = x^2$. Then for all $x, y \in \mathbb{R}^n$

$$\frac{1-\alpha^2}{2}\left(\langle x, Mx\rangle \langle y, My\rangle + \langle x, My\rangle^2\right) + \frac{\alpha^2}{\alpha^2 n + 1 - \alpha^2}\langle Mx, My\rangle^2 \geq \langle f(Mx), f(My)\rangle$$

with equality whenever $n = \binom{r+1}{2}$.

Proof: Define $\mathscr{W} : \mathbb{R}^n \to \mathscr{S}_r$ by $\mathscr{W} e_i = v_i v_i^\intercal$ and let $\mathscr{P} : \mathscr{S}_r \to \mathscr{S}_r$ be the orthogonal projection onto the span of $v_1 v_1^\intercal, \ldots, v_n v_n^\intercal$, so that $\mathscr{P} = \mathscr{W}(\mathscr{W}^\# \mathscr{W})^{-1} \mathscr{W}^\#$.

Observe that $\mathscr{W}^\# \mathscr{W} = (1-\alpha^2)I + \alpha^2 J$ is invertible, with $(\mathscr{W}^\# \mathscr{W})^{-1} = \frac{1}{1-\alpha^2}\left(I - \frac{\alpha^2}{\alpha^2 n + 1 - \alpha^2} J\right)$.

Define $X = \frac{1}{2}\left(Vx(Vy)^\intercal + Vy(Vx)^\intercal\right)$ and let $u = \mathscr{W}^\# X$. Then

$$\|X\|_F^2 \geq \|\mathscr{P} X\|_F^2 = X^\# \mathscr{P} X = X^\# \mathscr{W}(\mathscr{W}^\# \mathscr{W})^{-1} \mathscr{W}^\# X$$

$$= \frac{1}{1-\alpha^2} u^\intercal \left(I - \frac{\alpha^2}{\alpha^2 n + 1 - \alpha^2} J\right) u$$

Theorem(B.): Let $M = V^\intercal V$ be the Gram matrix of $v_1, \dots, v_n$ and let $f(x) = x^2$. Then for all $x, y \in \mathbb{R}^n$

$$\frac{1-\alpha^2}{2}\left(\langle x, Mx\rangle \langle y, My\rangle + \langle x, My\rangle^2\right) + \frac{\alpha^2}{\alpha^2 n + 1 - \alpha^2}\langle Mx, My\rangle^2 \geq \langle f(Mx), f(My)\rangle$$

with equality whenever $n = \binom{r+1}{2}$.

So we have $(1-\alpha^2)||X||_F^2 \geq u^\intercal u - \frac{\alpha^2}{\alpha^2 n+1-\alpha^2} u^\intercal J u$, where $X = \frac{1}{2}\left(Vx(Vy)^\intercal + Vy(Vx)^\intercal\right)$ and $u = \mathscr{W}^{\#} X$.

Now using the fact that $\langle ab^\intercal, cd^\intercal\rangle_F = \langle a, c\rangle \langle d, b\rangle$, we compute

$$\begin{aligned}\langle v_i v_i^\intercal, Vx(Vy)^\intercal\rangle_F = \langle Ve_i, Vx\rangle \langle Ve_i, Vy\rangle &= \langle e_i, V^\intercal V x\rangle \langle e_i, V^\intercal V y\rangle \\ &= (Mx)_i (My)_i.\end{aligned}$$

By symmetry, we conclude $u_i = \langle v_i v_i^\intercal, X\rangle_F = (Mx)_i (My)_i$, and thus $u^\intercal u = \sum_{i=1}^n (Mx)_i^2 (My)_i^2 = \langle f(Mx), f(My)\rangle$, as well as

$$u^\intercal J u = u^\intercal \mathbb{1}\mathbb{1}^\intercal u = \left(\sum_{i=1}^n u_i\right)^2 = \left(\sum_{i=1}^n (Mx)_i (My)_i\right)^2 = \langle Mx, My\rangle^2.$$

Theorem(B.): Let $M = V^{\intercal}V$ be the Gram matrix of $v_1, \dots, v_n$ and let $f(x) = x^2$. Then for all $x, y \in \mathbb{R}^n$

$$\frac{1-\alpha^2}{2}\left(\langle x, Mx\rangle \langle y, My\rangle + \langle x, My\rangle^2\right) + \frac{\alpha^2}{\alpha^2 n + 1 - \alpha^2}\langle Mx, My\rangle^2 \geq \langle f(Mx), f(My)\rangle$$

with equality whenever $n = \binom{r+1}{2}$.

So we have

$$(1-\alpha^2)||X||_F^2 \geq \langle f(Mx), f(My)\rangle - \frac{\alpha^2}{\alpha^2 n + 1 - \alpha^2}\langle Mx, My\rangle^2 .$$

It remains to verify that $||X||_F^2 = \frac{1}{2}\left(\langle x, Mx\rangle \langle y, My\rangle + \langle x, My\rangle^2\right)$, from which the desired inequality follows.

Moreover, recall that $\mathscr{S}_r$ has dimension $\binom{r+1}{2}$ and $v_1 v_1^{\intercal}, \dots, v_n v_n^{\intercal}$ are linearly independent. Therefore, if $n = \binom{r+1}{2}$ then $v_1 v_1^{\intercal}, \dots, v_n v_n^{\intercal}$ span $\mathscr{S}_r$, in which case $\mathscr{P}$ is the identity map and so we have equality above. $\square$

Theorem(B.): Let $M = V^\intercal V$ be the Gram matrix of $v_1, \dots, v_n$ and let $f(x) = x^2$. Then for all $x, y \in \mathbb{R}^n$

$$\frac{1-\alpha^2}{2}\left(\langle x, Mx\rangle \langle y, My\rangle + \langle x, My\rangle^2\right) + \frac{\alpha^2}{\alpha^2 n + 1 - \alpha^2}\langle Mx, My\rangle^2 \geq \langle f(Mx), f(My)\rangle$$

with equality whenever $n = \binom{r+1}{2}$.

Switching: For any unit vector v_i along the ith line, we can replace it with $-v_i$. This flips all edges and non-edges incident to v_i. Therefore, without loss of generality, we may switch $v_2, \dots, v_n$ so that $\langle v_i, v_1\rangle = \alpha$ for all $i \geq 2$.

Corollary(B.): For all $i \geq 2$, the degree $d(v_i)$ of the ith vertex of G satisfies

$$\left(n - 2d(v_i) + \tfrac{2}{\alpha} - 2\right)^2 \geq \left(n + \tfrac{1}{\alpha^2} - 1\right)\left(n - \tfrac{1}{2}\left(\tfrac{1}{\alpha^2} - 1\right)\left(\tfrac{1}{\alpha^2} - 3\right)\right),$$

with equality whenever $n = \binom{r+1}{2}$.

Proof: Apply the above theorem with $x = e_1$ and $y = e_i$. $\square$

Corollary(B.): For all $i \geq 2$, the degree $d(v_i)$ of the ith vertex of G satisfies
$\left(n - 2d(v_i) + \frac{2}{\alpha} - 2\right)^2 \geq \left(n + \frac{1}{\alpha^2} - 1\right)\left(n - \frac{1}{2}\left(\frac{1}{\alpha^2} - 1\right)\left(\frac{1}{\alpha^2} - 3\right)\right),$
with equality whenever $n = \binom{r+1}{2}$.

In particular, if $n > \frac{1}{2}\left(\frac{1}{\alpha^2} - 1\right)\left(\frac{1}{\alpha^2} - 3\right)$ then for each $i \geq 2$, we have either $d(v_i) < \frac{1}{4\alpha^4}$ or $d(v_i) > n - \frac{1}{4\alpha^4}$.

Observation: The Gram matrix M satisfies
$\frac{1}{2\alpha}M = \frac{1}{2}J - A + \frac{1-\alpha}{2\alpha}I$ where A is the adjacency matrix of G.

Since M is positive semidefinite

$$
\begin{aligned}
0 \leq \mathbb{1}^\intercal \frac{1}{2\alpha} M \mathbb{1} &= \frac{1}{2}\mathbb{1}^\intercal J \mathbb{1} - \mathbb{1}^\intercal A \mathbb{1} + \frac{1-\alpha}{2\alpha}\mathbb{1}^\intercal I \mathbb{1} \\
&= \frac{n^2}{2} - \sum_{i=1}^{n} d(v_i) + \frac{1-\alpha}{2\alpha} n,
\end{aligned}
$$

so that the average degree $\overline{d}$ of G is at most $\frac{n}{2} + \frac{1-\alpha}{2\alpha}$.

Therefore, letting $H = \{v_i : d(v_i) > n - \frac{1}{4\alpha^4}\}$ be the set of high degree vertices, we have $\frac{n}{2} + \frac{1-\alpha}{2\alpha} \geq \bar{d} \geq |H|\left(1 - \frac{1}{4\alpha^4 n}\right)$.

In particular, if $n \gg \frac{1}{\alpha^4}$ then $|H| \leq (1 + o(1))\frac{n}{2}$.

In this case, we can actually get a stronger bound by double counting the edges between H and its complement H^c:

$$(1 - o(1))|H|(n - |H|) \leq \sum_{v \in H} (d(v) - |H|) \leq e(H, H^c) \leq \sum_{v \in H^c} d(v) \leq \frac{n}{4\alpha^4}$$

Thus $|H| \leq \frac{1+o(1)}{2\alpha^4}$, and using this improved bound in the above, we actually conclude $|H| \leq \frac{1+o(1)}{4\alpha^4}$.

By switching all vertices in H, we obtain a graph with $\Delta \leq \frac{1+o(1)}{2\alpha^4}$.

Lemma(B.): If $n \gg \frac{1}{\alpha^4}$, then we can choose $v_1, \ldots, v_n$ such that the corresponding graph has max degree $\Delta \leq \frac{1+o(1)}{2\alpha^4}$.

Recalling that $A = \frac{1}{2}J + \frac{1-\alpha}{2\alpha}I - \frac{1}{2\alpha}M$, we have for any $y \perp \mathbb{1}$

$$y^\intercal A y = \frac{1-\alpha}{2\alpha} y^\intercal y - \frac{1}{2\alpha} y^\intercal M y \leq \frac{1-\alpha}{2\alpha} y^\intercal y.$$

Idea: As in the Alon-Boppana theorem, use a vertex and its neighbors to construct a vector $y \perp \mathbb{1}$ with $\frac{y^\intercal A y}{y^\intercal y}$ large.

Lemma(B.): If $n \gg \frac{1}{\alpha^4}$, then for all $t \leq \Delta$, $\frac{1+o(1)}{2\alpha} \geq \sqrt{t} - \frac{t\Delta}{n}$.

Proof: WLOG assume v_1 has $v_2, \ldots, v_{t+1}$ as neighbors and define $x \in \mathbb{R}^n$ by

$$x_i = \begin{cases} 1 & \text{if } i = 1 \\ 1/\sqrt{t} & \text{if } 2 \leq i \leq t+1 \\ 0 & \text{otherwise} \end{cases}$$

, so that $\frac{x^\intercal A x}{x^\intercal x} = \sqrt{t}$.

The result follows by taking $y = x - \frac{\langle x, \mathbb{1} \rangle}{n}\mathbb{1}$ to be the projection of x onto the orthogonal complement of $\mathbb{1}$. $\square$

Lemma(B.): If $n \gg \frac{1}{\alpha^4}$, then for all $t \le \Delta$, $\frac{1+o(1)}{2\alpha} \ge \sqrt{t} - \frac{t\Delta}{n}$.

Theorem(B.): $n \le (1+o(1)) \max\left(\frac{1}{\alpha^5}, 2r\right)$

Proof: WLOG suppose that $n \ge \frac{1}{\alpha^5} \gg \frac{1}{\alpha^4}$. If $n \le 2\Delta^{3/2}$, apply above lemma with $t = \frac{n^2}{4\Delta^2}$ to conclude $\frac{1+o(1)}{2\alpha} \ge \frac{n}{4\Delta}$, so that $n \le (1+o(1))\frac{2\Delta}{\alpha} \le \frac{1+o(1)}{\alpha^5}$.

Otherwise, $n > 2\Delta^{3/2}$ and so taking $t = \Delta$ in the above lemma gives $\frac{1+o(1)}{2\alpha} \ge \sqrt{\Delta}\left(1 - \frac{\Delta^{3/2}}{n}\right) > \frac{\sqrt{\Delta}}{2}$.

Thus $\Delta \le \frac{1+o(1)}{\alpha^2}$, so that $\frac{\Delta^{3/2}}{n} \le \frac{1+o(1)}{\alpha^3 n} = o(1)$ and thus applying the above inequality again we conclude $\Delta \le \frac{1+o(1)}{4\alpha^2}$.

Now consider $S = M - \alpha J$, and observe that $\mathrm{tr}(S) = (1-\alpha)n$ and $\mathrm{tr}(S^2) = \sum_{i=1}^n \left((1-\alpha)^2 + (-2\alpha)^2 d(v_i)\right) \le (2+o(1))n$.

Using $\mathrm{rk}(S) \le r+1$ and applying the eigenvalue Cauchy-Schwarz inequality $\mathrm{tr}(S)^2 \le \mathrm{rk}(S)\,\mathrm{tr}(S^2)$ implies the desired. $\square$

Future directions for research

- Unit vectors corresponding to equiangular lines are equivalently spherical $\{\alpha, -\alpha\}$-codes. Extend methods to all spherical L-codes with $|L| = s$ as well as $L = [-1, \alpha]$.

- Generalize to equiangular subspaces.

- Generalize to signed graphs and unitarily-signed graphs

- Apply methods to other graph matrices (ex: Laplacian).

- There is a conjecture on the multiplicity of the second eigenvalue of Laplacians on surfaces (due to Colin de Verdière). Prove a version of this conjecture for the adjacency matrix of (regular) graphs: $\mathrm{mult}(\lambda_2) \leq O(\sqrt{|E(G)|})$.


thank you
рахмат
danke
謝謝
ngiyabonga
teşekkür ederim
spas
Баярлалаа
mersi
kia ora
barka
welalin
tack
mahalo
tapadh leat
спасибо
faafetai lava
vinaka
спасибі
blagodaram
dank je
misaotra
matondo
paldies
grazzi
gracias
хвала
kiitos
dankie
nanni
nandri
dhanyavad
maururu
asante
manana
obrigada
tenki
bedankt
bayarlalaa
gràcie
hvala
köszönöm
akun
dankon
ačiū
djiere dieuf
tau
mochchakkeram
murakoze
enkosi
mamnun
дякую
dziękuję
chnorakaloutioun
gratias ago
gràcies
sulpáy
go raibh maith agat
chokrane
sobodi
dėkuji
sağolun
sukriya
kop khun krap
taiku
arigatô
takk
dakujem
trugarez
obrigado
mèsi
ありがとう
tanemirt
grazie
didi madloba
najis tuke
rahmet
shukriya
mercê
мерси
kam sah hamnida
rahmat
terima kasih
diolch
dhanyavadagalu
তোমাকে ধন্যবাদ
감사합니다
xièxie
ευχαριστώ
merci
תודה רבה