EQUIANGULAR LINES AND REGULAR GRAPHS

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Definition: A set of lines passing through the origin is called **equiangular**, if every pair of lines make the same angle.

Question: Determine $N(r)$, the maximum number of equiangular lines in $\mathbb{R}^r$.

Connections:
- Elliptic geometry
- Frame theory
- Theory of polytopes
- Spectral graph theory
- Algebraic number theory
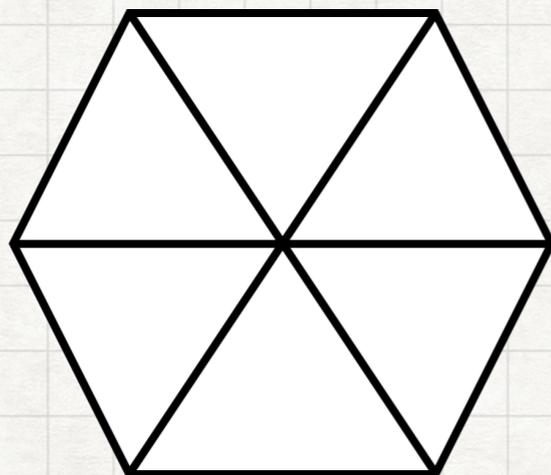- Quantum information theory

Earliest work:
Haantjes, Seidel 47-48

Blumenthal 49

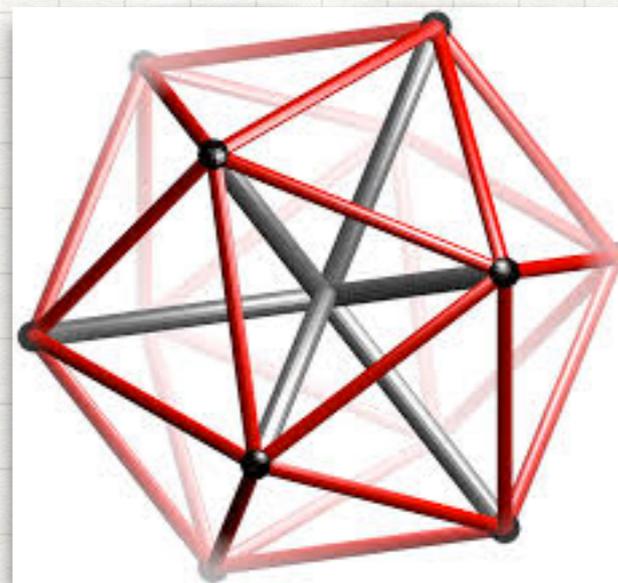Van Lint, Seidel 66

Lemmens, Seidel 73

…

Examples

r = 2: Regular Hexagon
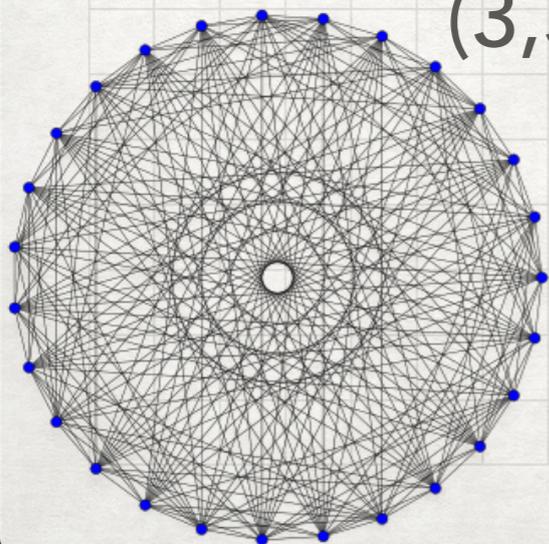
3 lines

r = 3: Regular Icosahedron

6 lines

r = 7:

28 lines

Take all 28 permutations of the vector (3,3,-1,-1,-1,-1,-1,-1).
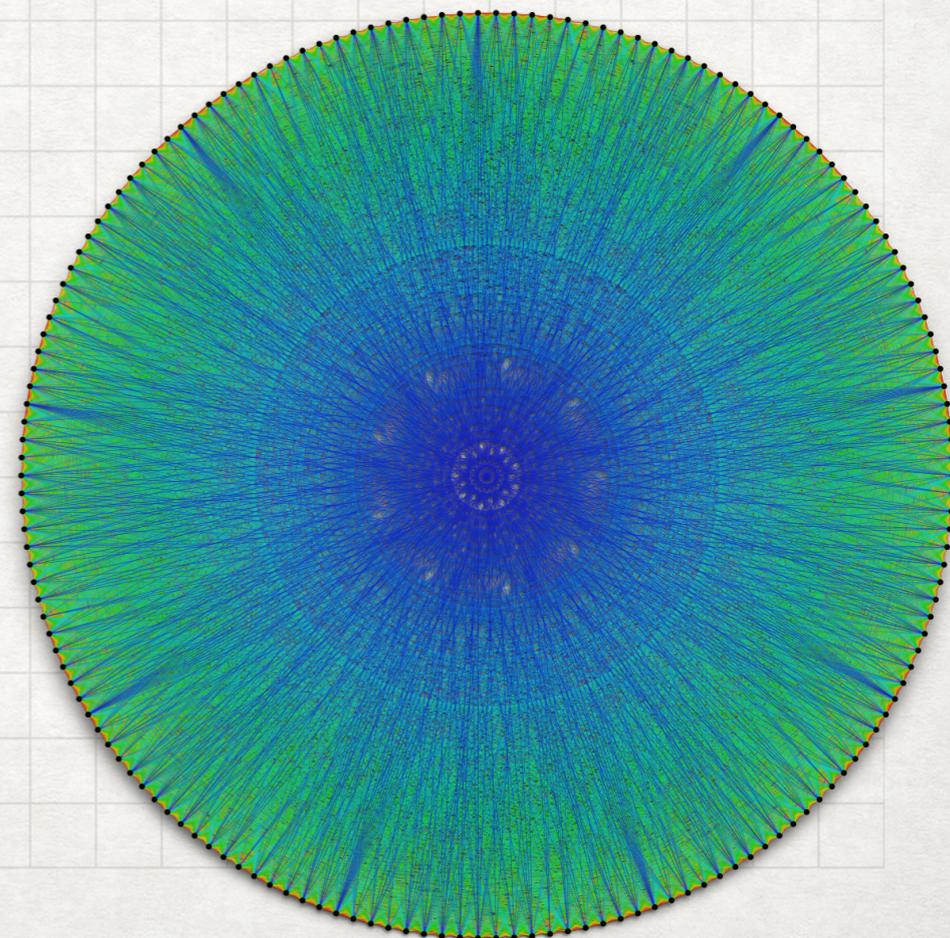
Schläfli Graph

r = 23:

276 lines

McLaughlin Graph

Theorem[Absolute bound] (Gerzon 73): $N(r) \leq \binom{r+1}{2}$

Proof: Let $v_1, \ldots, v_n$ be unit vectors along the given lines.
Then $\langle v_i, v_j \rangle = \pm \alpha$ for some $0 \leq \alpha < 1$.

Consider the matrices $v_1 v_1^{\mathsf{T}}, \ldots, v_n v_n^{\mathsf{T}}$. They live in the
$\binom{r+1}{2}$-dimensional space of symmetric matrices $\mathscr{S}_r$.

Recalling the Frobenius inner product of matrices
$$\langle A, B \rangle_F = \operatorname{tr}(A^{\mathsf{T}} B) = \sum_{i,j} A_{i,j} B_{i,j}$$

we have $\langle v_i v_i^{\mathsf{T}}, v_j v_j^{\mathsf{T}} \rangle_F = \operatorname{tr}(v_i v_i^{\mathsf{T}} v_j v_j^{\mathsf{T}}) = (v_i^{\mathsf{T}} v_j)^2 = \begin{cases} 1 & i = j \\ \alpha^2 & i \neq j \end{cases}$

Hence they are linearly independent. $\square$

What is known?

Theorem[Absolute bound] (Gerzon 73): $N(r) \leq \binom{r+1}{2}$.

- tight in dimension 2,3, 7 and 23. No other cases of equality are known.

Theorem (Neumann 73): If $n > 2r$ then $1/\alpha$ is an odd integer.

Question (Lemmens, Seidel 73):

Determine $N_\alpha(r)$, the maximum number of equiangular lines in $\mathbb{R}^r$ with common angle $\arccos(\alpha)$, especially when $\alpha = 1/3, 1/5, 1/7, \ldots$

Theorem[Relative Bound] (Lemmens, Seidel 73): $N_\alpha(r) \leq r\frac{1-\alpha^2}{1-r\alpha^2}$ for all $\alpha < \frac{1}{\sqrt{r}}$.

Recent progress

Theorem (B., Dräxler, Keevash, Sudakov 17): $N_\alpha(r) \leq 2r - 2$ for all $\alpha \gg \frac{1}{\sqrt{\log r}}$ with equality if and only if $\alpha = 1/3$.

Theorem (Jiang, Tidor, Yao, Zhang, Zhao 19): Let k_α be the minimum number of vertices in a graph with spectral radius $\frac{1-\alpha}{2\alpha}$. Then $N_\alpha(r) = \left\lfloor \frac{r-1}{1-1/k_\alpha} \right\rfloor$ for all $\alpha \gg \frac{k_\alpha}{\log \log r}$.

Question: What about for $\frac{1}{\sqrt{r}} \leq \alpha \leq O\left(\frac{1}{\sqrt{\log r}}\right)$?

Theorem (Yu 17): $N_\alpha(r) \leq \frac{1}{2}\left(\frac{1}{\alpha^2} - 1\right)\left(\frac{1}{\alpha^2} - 2\right)$ for all $\frac{1}{\sqrt{r+2}} \leq \alpha \leq \sqrt{\frac{3}{r+16}}$.

Theorem (Glazyrin, Yu 18): $N_\alpha(r) \leq \left(\frac{2}{3\alpha^2} + \frac{4}{7}\right)r + 2$ for all $\alpha \leq \frac{1}{3}$.
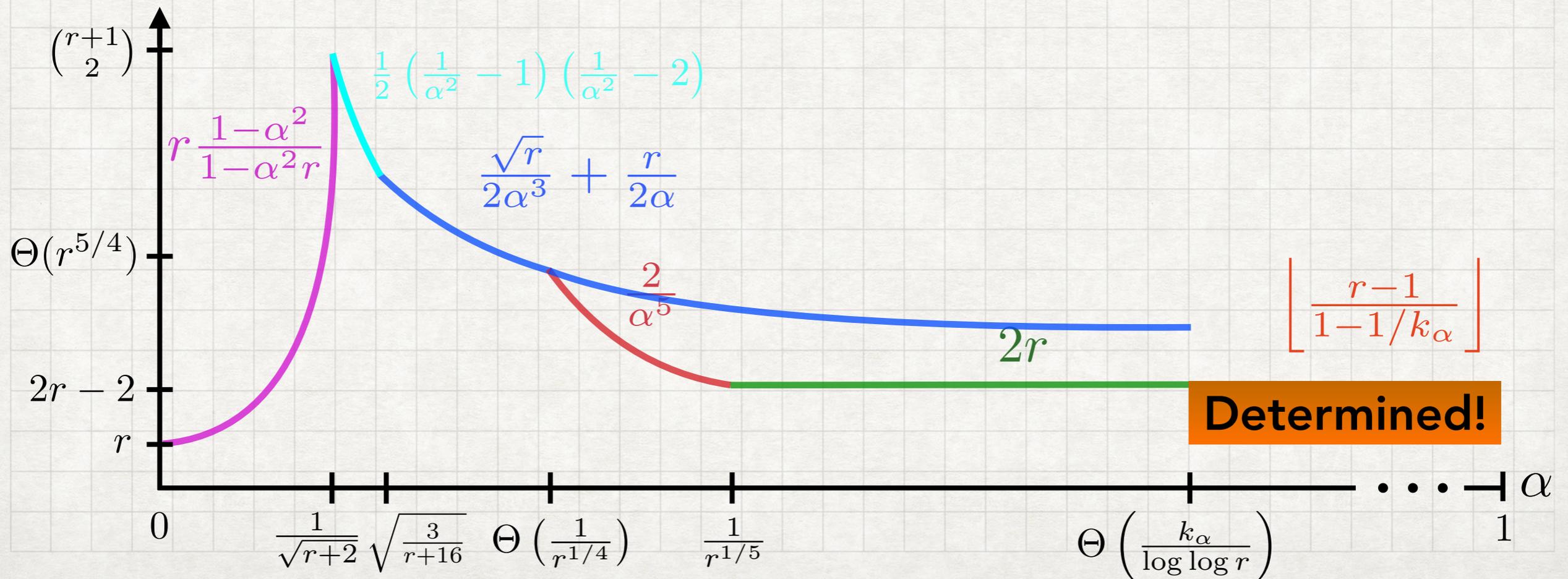
New results

Theorem(B.): $N_\alpha(r) \le \frac{\sqrt{r}}{2\alpha^3} + \frac{(1+\alpha)r}{2\alpha}$.

- Based on bounding the largest eigenvalue of a Gram matrix.

Theorem(B.): $N_\alpha(r) \le \max\left(\frac{2}{\alpha^5} + \frac{2}{\alpha^3(1-\alpha)}, \left(2 + \frac{8\alpha^2}{(1-\alpha)^2}\right)(r+1)\right)$

$$= (1 + o(1))\max\left(\frac{2}{\alpha^5}, 2r\right) .$$

- Based on bounding degrees of a graph, bootstrapped via an Alon-Boppana theorem.

New results for regular graphs

Corollary(B.): Let G be a k-regular graph with second and last eigenvalue λ_2, λ_n. If the spectral gap satisfies $k - \lambda_2 \ll n$, then

$$\lambda_2 \geq (1 - o(1))k^{1/3} \quad \text{and} \quad \lambda_2 \geq (1 - o(1))\sqrt{-\lambda_n}.$$

Theorem(B.): If G is a k-regular graph with $k - \lambda_2 < \frac{n}{2}$, then

$$2\left(k - \frac{(k-\lambda_2)^2}{n}\right) \leq \frac{\lambda_2(\lambda_2+1)(2\lambda_2+1)}{1-\frac{2(k-\lambda_2)}{n}} - \lambda_2(3\lambda_2 + 1),$$

$$-\lambda_n \leq \frac{\lambda_2(\lambda_2+1)}{1-\frac{2(k-\lambda_2)}{n}} - \lambda_2,$$

with equality in both whenever $n + 1 = \binom{n-\text{mult}(\lambda_2)+1}{2}$, i.e. when G corresponds to a set of real equiangular lines meeting the absolute bound in dimension $r = n - \text{mult}(\lambda_2)$.

Corollary(B.): Let G be a k-regular graph with second eigenvalue λ_2. If the spectral gap satisfies $k - \lambda_2 \ll n$, then

$$\lambda_2 \geq (1 - o(1))k^{1/3} \ .$$

Proof sketch: Starting with the adjacency matrix A, let $\alpha = \frac{1}{2\lambda_2 + 1}$ and define $M = (1 - \alpha)I + \alpha J - 2\alpha A$.

Straightforward to check that M is positive semidefinite, so it is the Gram matrix of some unit vectors $v_1, \ldots, v_n$.

Note that $\overline{v} = \frac{1}{n} \sum_{i=1}^n v_i$ is equidistant from each v_i.

Project $X = \overline{v}v_1^\mathsf{T} + v_1\overline{v}^\mathsf{T}$ onto the span of $v_1 v_1^\mathsf{T}, \ldots, v_n v_n^\mathsf{T}$ (orthogonally with respect to the Frobenius inner product).

The (Frobenius) norm of X can only decrease! $\quad\square$

New results in the complex setting

Given a pair of complex lines $U, V \subset \mathbb{C}^r$, the quantity $|\langle u, v \rangle|$ is the same for all unit vectors $u \in U, v \in V$ and so $\arccos|\langle u, v \rangle|$ is called the **Hermitian angle** between U and V.

We define $N_\alpha^{\mathbb{C}}(r)$ to be the maximum number of complex equiangular lines in $\mathbb{C}^r$ with common Hermitian angle $\arccos(\alpha)$.

Theorem[Absolute bound] (Delsarte, Goethals, Seidel 75): $N_\alpha^{\mathbb{C}}(r) \leq r^2$

Conjecture (Zauner 99): For each $r \in \mathbb{N}$, $\max_\alpha N_\alpha^{\mathbb{C}}(r) = r^2$ and a construction can be obtained as the orbit of a vector under the action of a Weyl-Heisenberg group.

Collections of r^2 complex equiangular lines in $\mathbb{C}^r$ are known as SICs/SIC-POVMs in quantum information theory.

New results in the complex setting

Theorem[Relative Bound] (Delsarte, Goethals, Seidel 75)**:**
$N_\alpha^{\mathbb{C}}(r) \leq r \frac{1-\alpha^2}{1-r\alpha^2}$ for all $\alpha < \frac{1}{\sqrt{r}}$.

Theorem(B.)**:** $N_\alpha^{\mathbb{C}}(r) \leq \frac{\sqrt{r}}{\alpha^3} + \frac{(1+\alpha)r}{\alpha}$.

Definition: Given vectors $v_1, \ldots, v_n$, let H be the $n \times n$ matrix defined by $H_{i,j} = |\langle v_i, v_j \rangle|^2$ and let $H^\dagger$ be its Moore-Penrose generalized inverse.

Theorem[Improved Welch bound](B.)**:** Given unit vectors $v_1, \ldots, v_n \in \mathbb{C}^r$, we have $\sum_{i,j} H_{i,j}^\dagger \leq r$ and moreover,

$$\sum_{i,j} |\langle v_i, v_j \rangle|^2 \geq \frac{n^2}{r} \left(2 - \frac{1}{r} \sum_{i,j} H_{i,j}^\dagger \right).$$

Future directions for research

- Unit vectors corresponding to equiangular lines are equivalently spherical $\{\alpha, -\alpha\}$-codes. Extend methods to more general spherical L-codes.

- Generalize to equiangular subspaces.

- Generalize to signed graphs and unitarily-signed graphs.

- Apply methods to other graph matrices (ex: Laplacian).