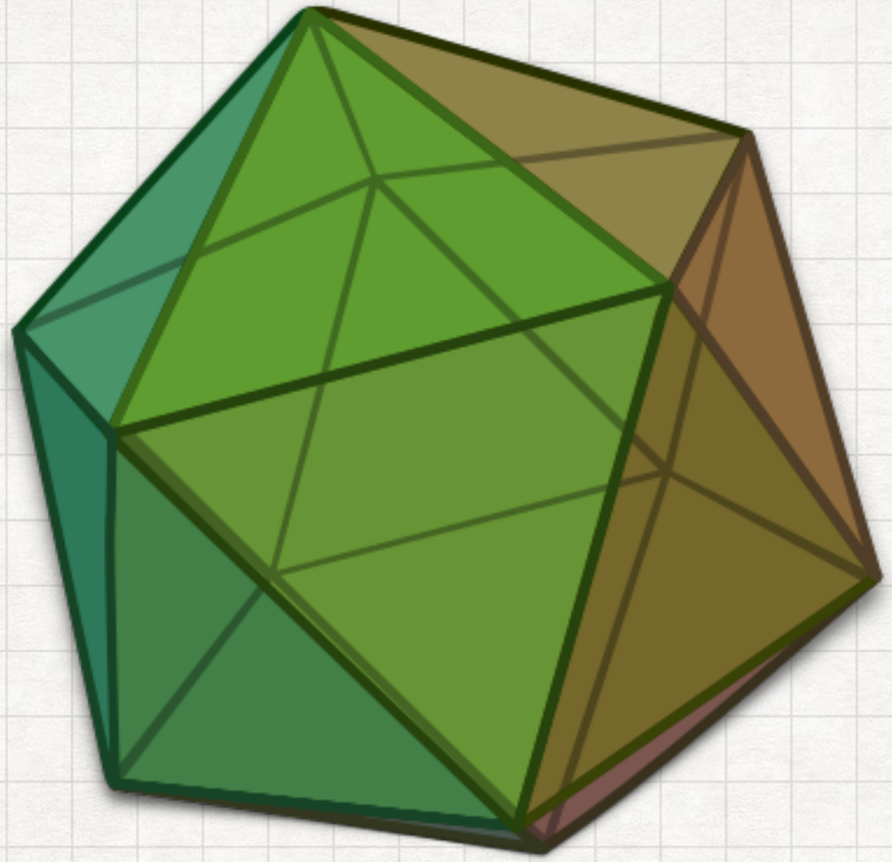

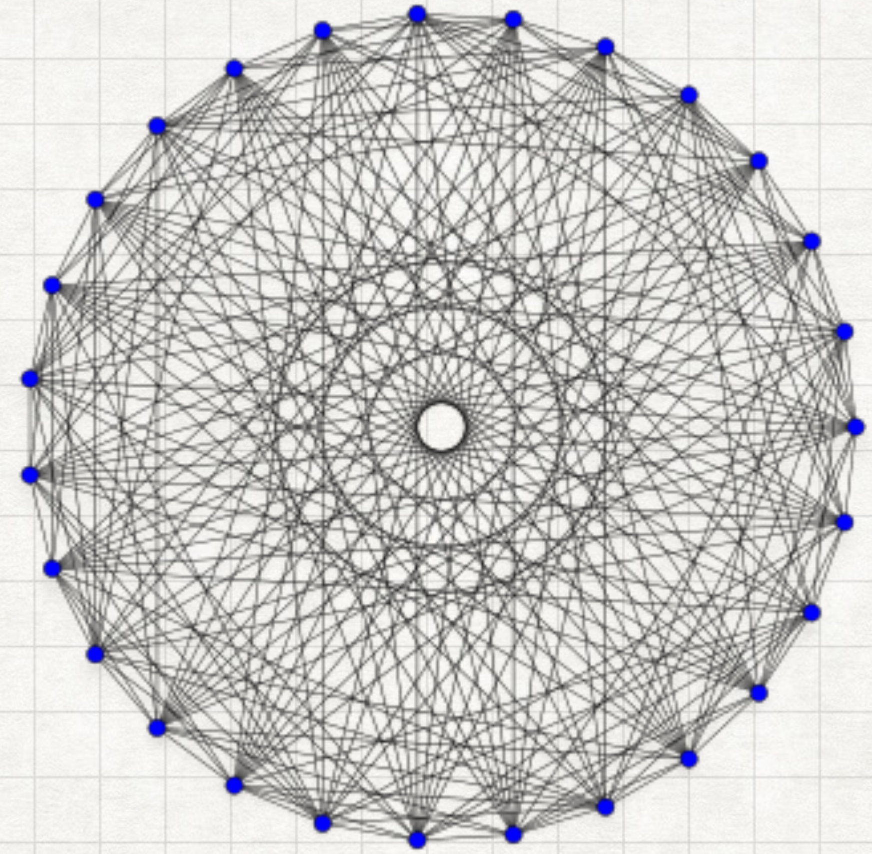

EQUIANGULAR LINES VIA IMPROVED EIGENVALUE MULTIPLICITY

By: Igor Balla, joint work with Matija Bucić

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Connections:

- Elliptic geometry
- Frame theory
- Theory of polytopes
- Banach space theory
- Spectral graph theory
- Algebraic number theory
- Quantum information theory

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Theorem(B., Bucić): For any positive integer k, if $r \geq 2^{\Omega(k^{20})}$ then

$$N_{\frac{1}{2k-1}}(r) = \left\lfloor \frac{r-1}{1-1/k} \right\rfloor.$$

Connection to graphs

Given a family of n equiangular lines in $\mathbb{R}^r$ with common angle $\arccos(\alpha)$, we can pick a unit vector along each line to get vectors $v_1, \ldots, v_n$ satisfying $\langle v_i, v_j \rangle = \pm\alpha$ for all $i \neq j$.

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If $n \geq r + 2$, then its second largest eigenvalue is $\lambda_2 = \frac{1}{2}\left(\frac{1}{\alpha} - 1\right)$ and has multiplicity at least $n - r - 1$.

Multiplicity of the second eigenvalue of a graph

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Moreover, if $n \geq 2^{\Delta^{\Omega(1)}}$, then $m(\lambda_2) \leq \frac{n}{\lambda_2+1} + n^{o(1)}$.

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If the graph is connected with min degree δ, this can be improved to $\frac{n}{\lambda_2 + \Omega(\delta \log_\Delta \log n)}$

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In each such induced subgraph, the number of **small** walks starting at v_i is at most ℓ^{2S}.

Idea: Count closed walks of length $2S$ before and after deleting some vertices uniformly at random!

The number of such walks starting at v_i is

$$e_i^\intercal A^{2S} e_i = e_i^\intercal A^{2S}_{B^S(v)} e_i \leq \lambda_1(B^S(v_i))^{2S} \leq \lambda_2^{2S}.$$

Let ℓ be a parameter and call a closed walk of length $2S$ **small** if its support has size at most ℓ. Otherwise call it **large**.

The number of closed walks of length 2ℓ starting at v_i is $\leq \lambda_2^{2\ell}$, so the number of connected induced subgraphs on ℓ vertices that contain v_i is at most $\lambda_2^{2\ell}$.

In each such induced subgraph, the number of **small** walks starting at v_i is at most ℓ^{2S}.

Thus the number of **small** walks starting at v_i is at most $\lambda_2^{2\ell}\ell^{2S}$!

Hence the total number of **small** walks (starting at some vertex) is at most $n\lambda_2^{2\ell}\ell^{2S}$.

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Now, if we choose $t = n\frac{\log \ell}{\ell}$ vertices uniformly at random, then for any **large** walk, the probability that we do not sample any of its vertices is $\binom{n-\ell}{t} \Big/ \binom{n}{t} \leq \left(1 - \frac{\ell}{n}\right)^t \leq \frac{1}{\ell}$.

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Since the total number of **large** walks is at most $n\lambda_2^{2S}$, with positive probability, there exist t vertices whose removal yields a subgraph H with at most $n\lambda_2^{2S}/\ell$ **large** walks.

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Thus H has at most $n\lambda_2^{2S}/\ell + n\lambda_2^{2\ell}\ell^{2S}$ closed walks of length $2S$ starting at some vertex.

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On the other hand, the number of such walks is at least

$$\operatorname{tr}(A_H^{2S}) \ge \lambda_2^{2S} m_H(\lambda_2).$$

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Multiplicity of λ_2 in H

Putting it all together and using the Cauchy interlacing theorem, we conclude that

$$m(\lambda_2) \leq t + m_H(\lambda_2) \leq n\frac{\log \ell}{\ell} + n\lambda_2^{2\ell-2S}\ell^{2S} + \frac{n}{\ell}.$$

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If $\lambda_2 \log \lambda_2 \ge \log_\Delta n$, then we let $\ell = \frac{S}{2\log S}$.

Otherwise if $\lambda_2 \log \lambda_2 < \log_\Delta n$, then we let $\ell = \lambda_2/3$. $\square$

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Open question: Is there a constant $C > 0$ such that if $n \geq \Delta^C$, then $m(\lambda_2) \lesssim \frac{n}{\lambda_2}$?

рахмат
danke
謝謝
ngiyabonga
spas
teşekkür ederim
Баярлалаа
mersi
kia ora
barka
welalin
tack
mahalo
tapadh leat
спасибо
faafetai lava
vinaka
спасибі
blagodaram
dank je
misaotra
matondo
paldies
grazzi
kiitos
dankie
nanni
nandri
dhanyavad
maururu
köszönöm
thank you
gracias
хвала
asante
manana
obrigada
tenki
bedankt
bayarlalaa
gràçie
hvala
akun
dankon
aćiū
djiere dieuf
tau
mochchakkeram
murakoze
enkosi
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sobodi
dēkuji
sagolun
sukriya
kop khun krap
taiku
obrigado
mèsi
didi madloba
najis tuke
ありがとう
tanemirt
grazie
arigatô
takk
dakujem
trugarez
kam sah hamnida
rahmat
terima kasih
rahmet
ευχαριστώ
diolch
dhanyavadagalu
shukriya
mercê
мерси
তোমাকে ধন্যবাদ
감사합니다
xièxie
merci
תודה רבה