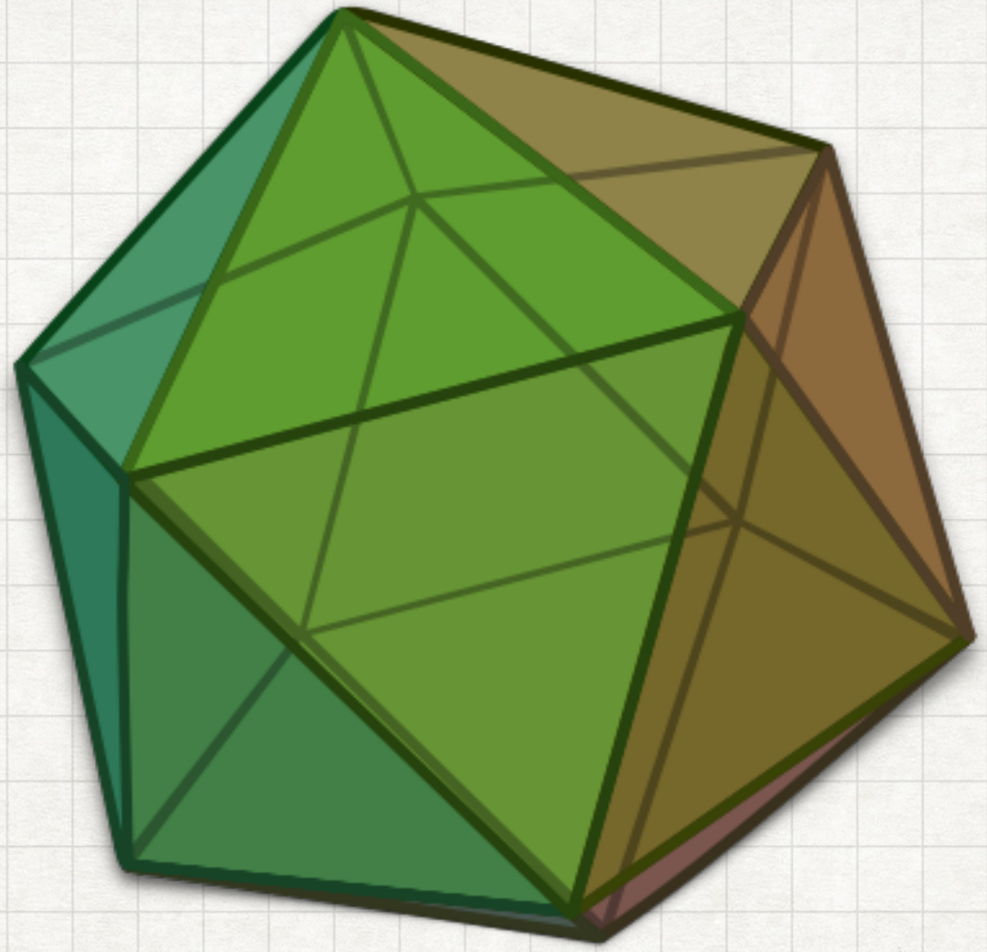
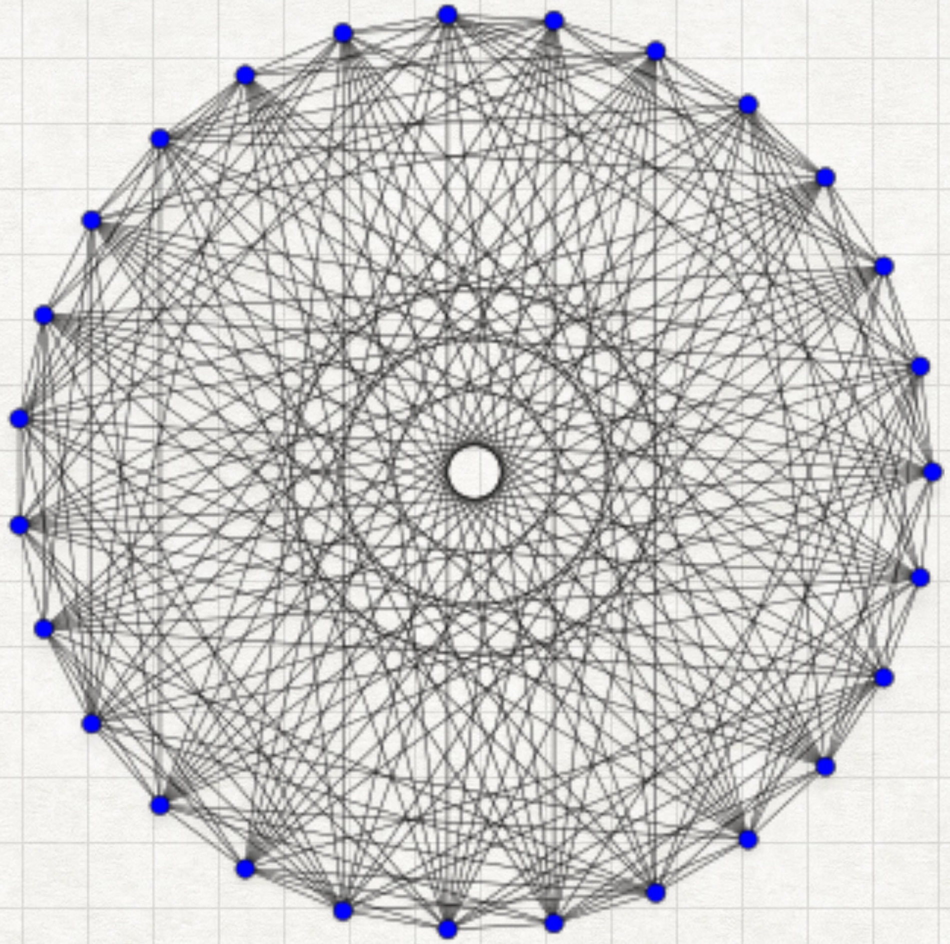




EQUIANGULAR LINES VIA MATRIX PROJECTION

By: Igor Balla

Definition: A set of lines passing through the origin is called **equiangular**, if every pair of lines make the same angle.

Question: Determine $N(r)$, the maximum number of equiangular lines in $\mathbb{R}^r$.

Connections:

- Elliptic geometry
- Frame theory
- Theory of polytopes
- Banach space theory
- Spectral graph theory
- Algebraic number theory
- Quantum information theory

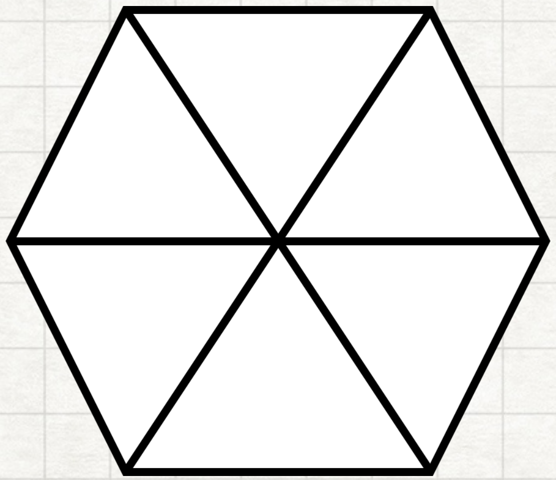
Earliest work:

Haantjes, Seidel 47-48
Blumenthal 49
Van Lint, Seidel 66
Lemmens, Seidel 73
…

Examples

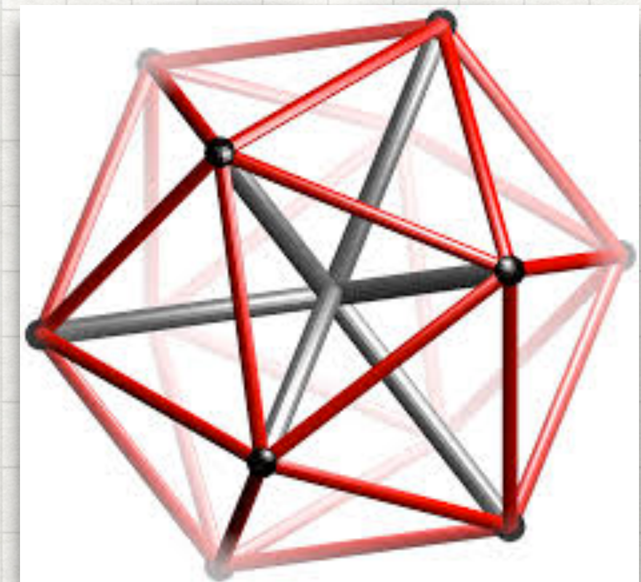
r = 2: Regular Hexagon

3 lines



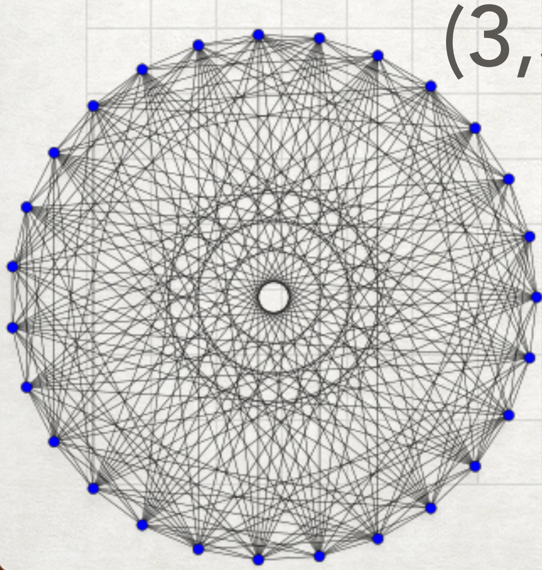
r = 3: Regular Icosahedron

6 lines



r = 7: Take all 28 permutations of the vector (3,3,-1,-1,-1,-1,-1,-1).

28 lines

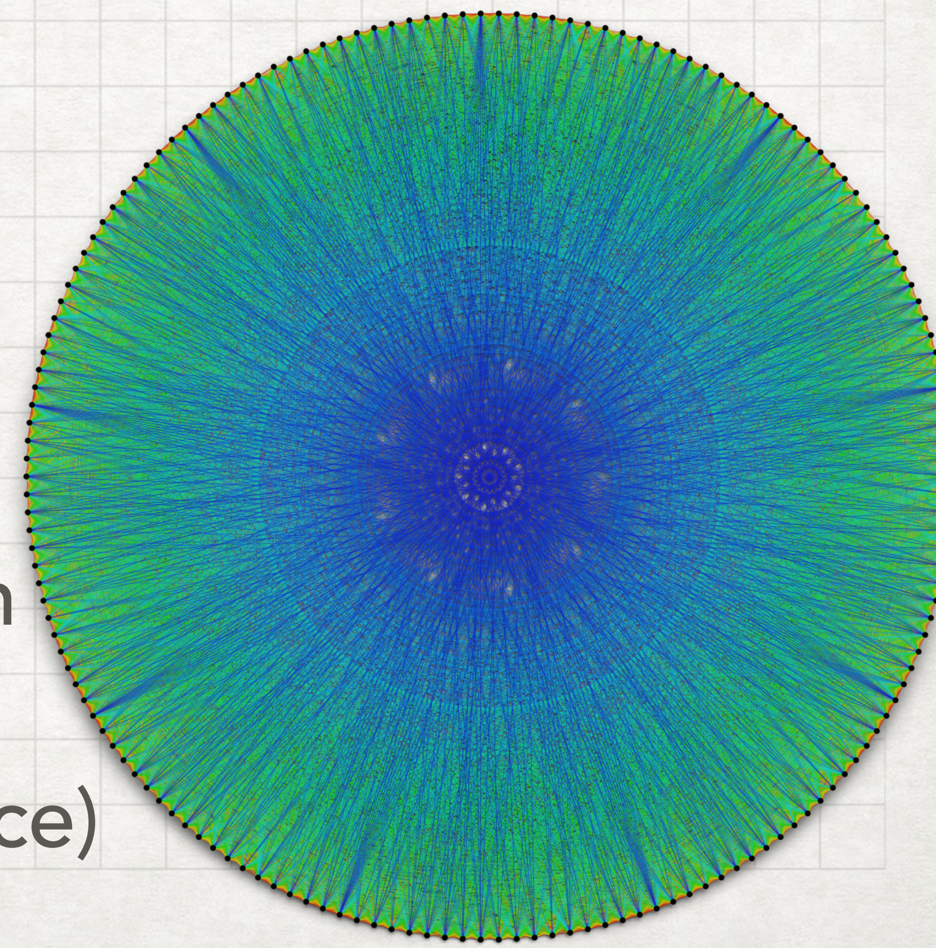


Schläfli Graph (E8 lattice)

r = 23:

276 lines

McLaughlin Graph (Leech lattice)



Theorem[Absolute bound] (Gerzon 73): $N(r) \leq \binom{r+1}{2}$

Proof: Let $v_1, \ldots, v_n$ be unit vectors along the given lines. Then $\langle v_i, v_j \rangle = \pm\alpha$ for some $0 \leq \alpha < 1$.

Consider the matrices $v_1 v_1^\intercal, \ldots, v_n v_n^\intercal$. They live in the $\binom{r+1}{2}$-dimensional space of symmetric matrices $\mathscr{S}_r$.

Recalling the Frobenius inner product of matrices

$$\langle A, B \rangle_F = \operatorname{tr}(A^\intercal B) = \sum_{i,j} A_{i,j} B_{i,j}$$

we have $\left\langle v_i v_i^\intercal, v_j v_j^\intercal \right\rangle_F = \operatorname{tr}(v_i v_i^\intercal v_j v_j^\intercal) = (v_i^\intercal v_j)^2 = \begin{cases} 1 & i = j \\ \alpha^2 & i \neq j \end{cases}$

Hence they are linearly independent. $\square$

What is known?

Theorem[Absolute bound] (Gerzon 73): $N(r) \leq \binom{r+1}{2}$.

- tight in dimension 2,3, 7 and 23. No other cases of equality are known.

Theorem (de Caen 00): $N(r) \geq \Omega(r^2)$

Theorem (Neumann 73): If $n > 2r$ then $1/\alpha$ is an odd integer.

Question (Lemmens, Seidel 73):
Determine $N_\alpha(r)$, the maximum number of equiangular lines in $\mathbb{R}^r$ with common angle $\arccos(\alpha)$, especially when $\alpha = 1/3, 1/5, 1/7, \ldots$

Theorem[Relative Bound] (Lemmens, Seidel 73): $N_\alpha(r) \leq r\frac{1-\alpha^2}{1-r\alpha^2}$ for all $r \leq 1/\alpha^2 - 2$.

Recent progress

Theorem (B., Dräxler, Keevash, Sudakov 17): $N_\alpha(r) \leq 2r-2$ if r is exponentially large in $1/\alpha^2$, with equality if and only if $\alpha = 1/3$.

Theorem (Jiang, Tidor, Yao, Zhang, Zhao 19): Let k_α be the minimum number of vertices in a graph with spectral radius $\frac{1-\alpha}{2\alpha}$. If r is doubly exponentially large in k_α/α, then

$$N_\alpha(r) = \left\lfloor \frac{r-1}{1-1/k_\alpha} \right\rfloor$$

Question: What about for $1/\alpha^2 - 2 \leq r \leq O(2^{1/\alpha^2})$?

Theorem (Yu 17): $N_\alpha(r) \leq \binom{1/\alpha^2-1}{2}$ for $1/\alpha^2 - 2 \leq r \leq 3/\alpha^2 - 16$

Theorem (Glazyrin, Yu 18): $N_\alpha(r) \leq \left(\frac{2}{3\alpha^2} + \frac{4}{7}\right) r + 2$ for all $\alpha \leq \frac{1}{3}$.

New results

Theorem(B.): There exists a positive constant C such that

equality iff the absolute bound is met in dimension $1/\alpha^2 - 2$

$$N_\alpha(r) \le \begin{cases} \binom{1/\alpha^2-1}{2} & \text{if } \frac{1}{\alpha^2} - 2 < r \le \frac{1-o(1)}{4\alpha^4} \\ (2+o(1))r & \text{if } \frac{1-o(1)}{4\alpha^4} < r \le O\left(\frac{1}{\alpha^5}\right) \\ \left(1 + \frac{1+o(1)}{4\cos^2\left(\frac{\pi}{q+2}\right)}\right) r & \text{if } \frac{1}{\alpha^{2q+1}} \ll r \le O\left(\frac{1}{\alpha^{2q+3}}\right) \text{ for integer } q \ge 2 \\ \left(\frac{5}{4} + o(1)\right) r & \text{if } 1/\alpha^{\omega(1)} \le r < 2^{1/\alpha^{4C}} \\ \left(1 + \frac{C\log(1/\alpha)}{\log\log r}\right) r & \text{if } 2^{1/\alpha^{4C}} \le r < 2^{1/\alpha^{C(k_\alpha - 1)}} \\ \left\lfloor \frac{r-1}{1-1/k_\alpha} \right\rfloor & \text{if } 2^{1/\alpha^{C(k_\alpha-1)}} \le r. \end{cases}$$

always equality!

Simple lower bounds: $N_\alpha(r) \ge r$ for all α, r, and if $k_\alpha < \infty$, then $N_\alpha(r) \ge \left\lfloor \frac{r-1}{1-1/k_\alpha} \right\rfloor$

New results for regular graphs

Corollary(B.): Let G be a k-regular graph with second and last eigenvalue λ_2, λ_n. If the spectral gap satisfies $k - \lambda_2 \ll n$, then

$$\lambda_2 \geq (1 - o(1))k^{1/3} \qquad \text{and} \qquad \lambda_2 \geq (1 - o(1))\sqrt{-\lambda_n}.$$

Theorem(B.): If G is a k-regular graph with $k - \lambda_2 < \frac{n}{2}$, then

$$2\left(k - \frac{(k-\lambda_2)^2}{n}\right) \leq \frac{\lambda_2(\lambda_2+1)(2\lambda_2+1)}{1 - \frac{2(k-\lambda_2)}{n}} - \lambda_2(3\lambda_2 + 1),$$

$$-\lambda_n \leq \frac{\lambda_2(\lambda_2+1)}{1 - \frac{2(k-\lambda_2)}{n}} - \lambda_2,$$

with equality in both whenever $n + 1 = \binom{n - \mathrm{mult}(\lambda_2) + 1}{2}$, i.e. when G corresponds to a set of real equiangular lines meeting the absolute bound in dimension $r = n - \mathrm{mult}(\lambda_2)$.

Corollary(B.): Let G be a k-regular graph with second eigenvalue λ_2. If the spectral gap satisfies $k - \lambda_2 \ll n$, then

$$\lambda_2 \geq (1 - o(1))k^{1/3} \ .$$

Proof sketch: Starting with the adjacency matrix A, let $\alpha = \frac{1}{2\lambda_2+1}$ and define $M = (1-\alpha)I + \alpha J - 2\alpha A$.

Straightforward to check that M is positive semidefinite, so it is the Gram matrix of some unit vectors $v_1, \ldots, v_n$.

Note that $\overline{v} = \frac{1}{n}\sum_{i=1}^{n} v_i$ is equidistant from each v_i.

Project $X = \overline{v}v_1^\intercal + v_1\overline{v}^\intercal$ onto the span of $v_1 v_1^\intercal, \ldots, v_n v_n^\intercal$ (orthogonally with respect to the Frobenius inner product).

The (Frobenius) norm of X can only decrease! $\square$

New results in the complex setting

Given a pair of complex lines $U, V \subset \mathbb{C}^r$, the quantity $|\langle u, v \rangle|$ is the same for all unit vectors $u \in U, v \in V$ and so $\arccos |\langle u, v \rangle|$ is called the **Hermitian angle** between U and V.

We define $N_\alpha^{\mathbb{C}}(r)$ to be the maximum number of complex equiangular lines in $\mathbb{C}^r$ with common Hermitian angle $\arccos(\alpha)$.

Theorem[Absolute bound] (Delsarte, Goethals, Seidel 75): $N_\alpha^{\mathbb{C}}(r) \leq r^2$

Conjecture (Zauner 99): For each $r \in \mathbb{N}$, $\max_\alpha N_\alpha^{\mathbb{C}}(r) = r^2$ and a construction can be obtained as the orbit of a vector under the action of a Weyl-Heisenberg group.

Collections of r^2 complex equiangular lines in $\mathbb{C}^r$ are known as SICs/SIC-POVMs in quantum information theory.

New results in the complex setting

Theorem[Relative Bound] (Delsarte, Goethals, Seidel 75)**:**
$N_\alpha^{\mathbb{C}}(r) \leq r\,\frac{1-\alpha^2}{1-r\alpha^2}$ for all $r \leq 1/\alpha^2 - 1\,.$

Theorem(B.)**:** If $r \leq \frac{1-o(1)}{\alpha^3}$, then $N_\alpha^{\mathbb{C}}(r) \leq \left(\frac{1}{\alpha^2} - 1\right)^2$, with equality if and only if there exists a SIC in $1/\alpha^2 - 1$ dimensions. Otherwise if $r > \frac{1-o(1)}{\alpha^3}$, then $N_\alpha^{\mathbb{C}}(r) \leq \frac{1+\alpha}{\alpha} r + O\left(\frac{1}{\alpha^3}\right)$.

Future directions for research

- Unit vectors corresponding to equiangular lines are equivalently spherical $\{\alpha, -\alpha\}$-codes. Extend methods to more general spherical L-codes.

- Generalize to equiangular subspaces.

- Generalize to signed graphs and unitarily-signed graphs.

- Apply methods to other graph matrices (ex: Laplacian).


thank you
danke
рахмат
謝謝
ngiyabonga
teşekkür ederim
Баярлалаа
спасибо
mersi
kia ora
barka
welalin
tack
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faafetai lava
vinaka
спасибі
blagodaram
dank je
misaotra
matondo
paldies
grazzi
mahalo
tapadh leat
gracias
хвала
asante
manana
kiitos
dankie
nanni
nandri
dhanyavad
maururu
köszönöm
ačiū
dankon
akun
obrigada
tenki
murakoze
bedankt
hvala
bayarlalaa
gràçie
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