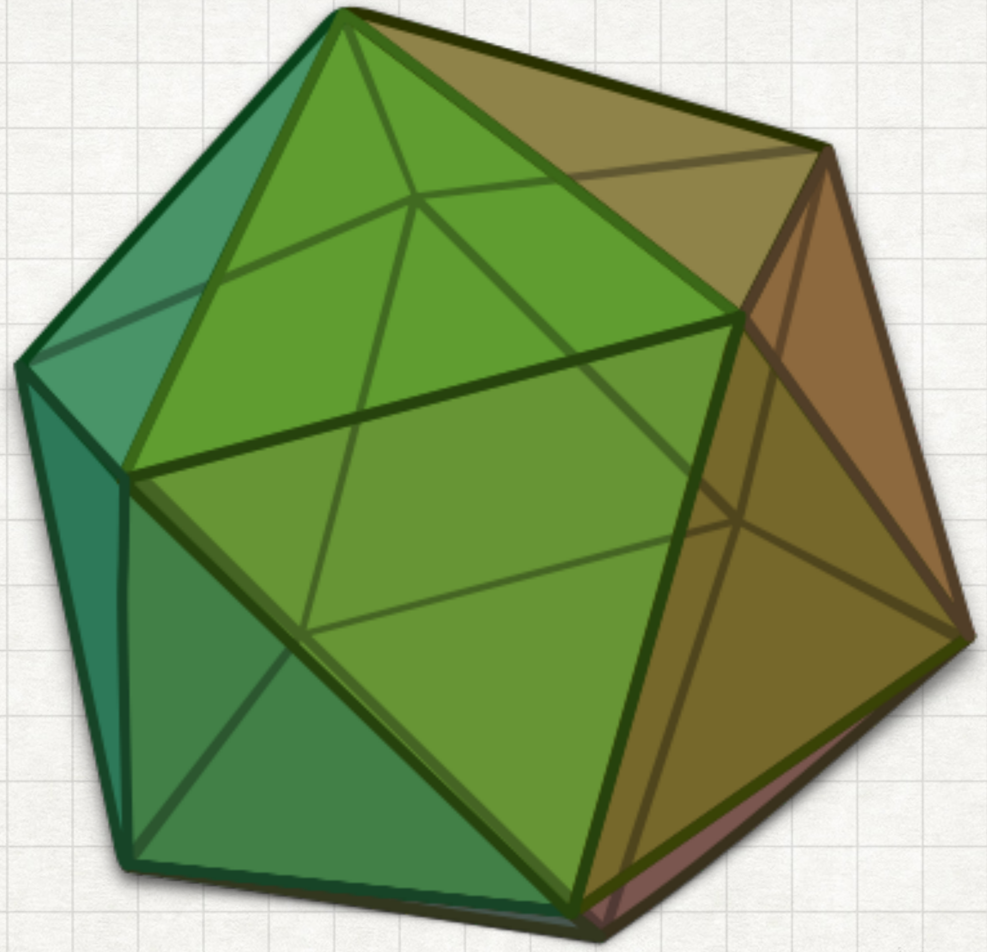
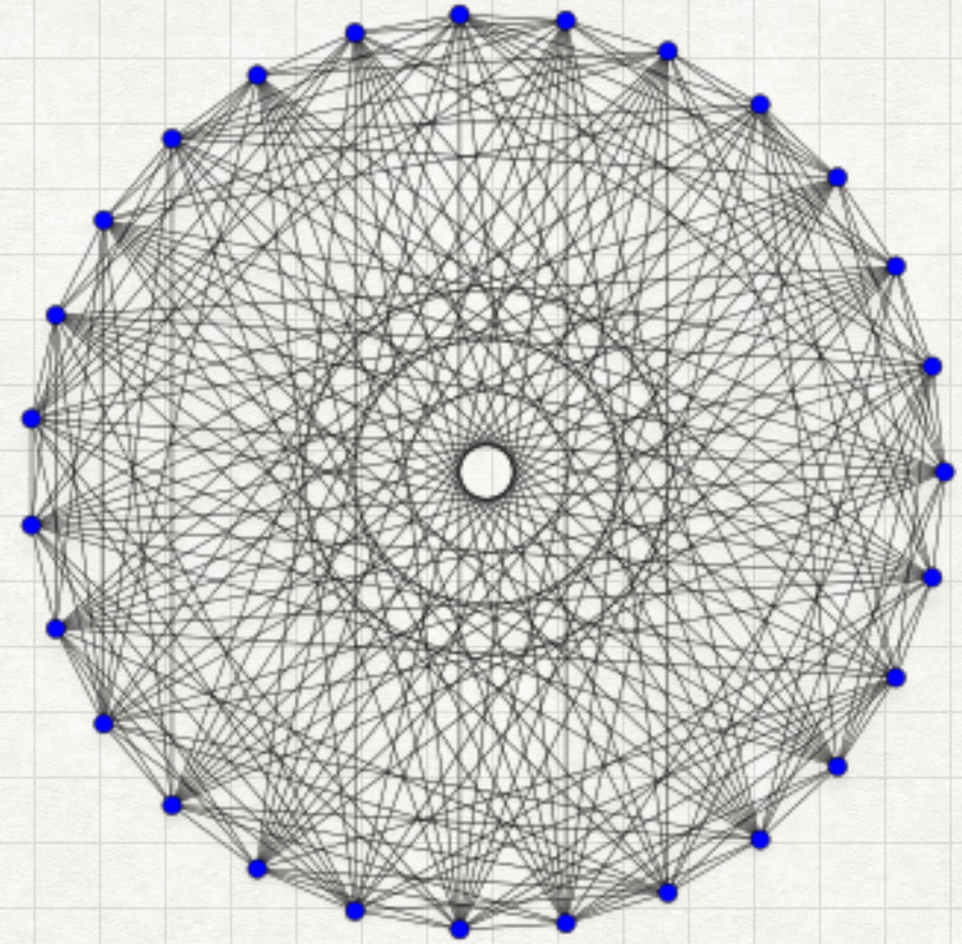




EQUIANGULAR LINES VIA MATRIX PROJECTION

By: Igor Balla

Definition: A set of lines passing through the origin is called **equiangular**, if every pair of lines make the same angle.

Question: Determine $N(r)$, the maximum number of equiangular lines in $\mathbb{R}^r$.

Connections:

- Elliptic geometry
- Frame theory
- Theory of polytopes
- Banach space theory
- Spectral graph theory
- Algebraic number theory
- Quantum information theory

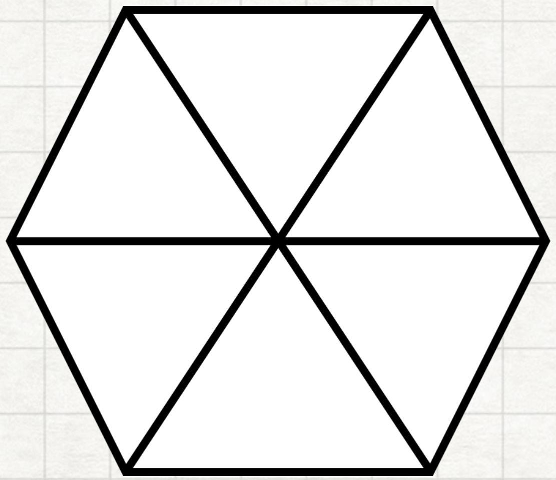
Earliest work:

Haantjes, Seidel 47-48
Blumenthal 49
Van Lint, Seidel 66
Lemmens, Seidel 73
…

Examples

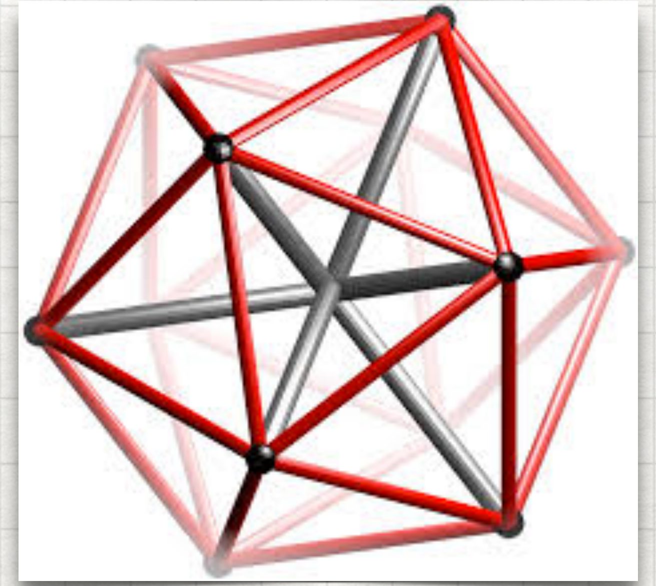
r = 2: Regular Hexagon

3 lines



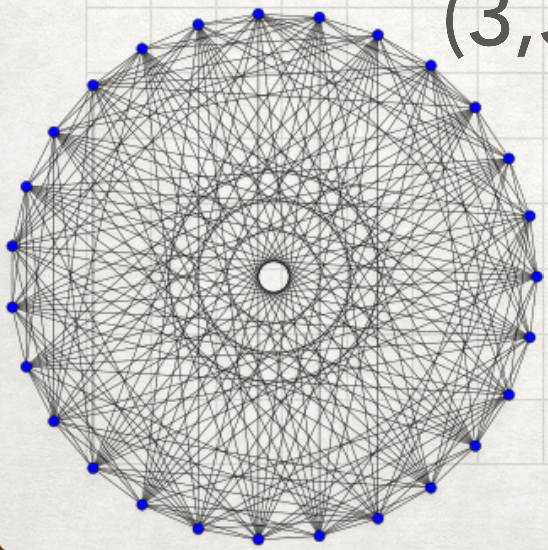
r = 3: Regular Icosahedron

6 lines



r = 7: Take all 28 permutations of the vector (3,3,-1,-1,-1,-1,-1,-1).

28 lines

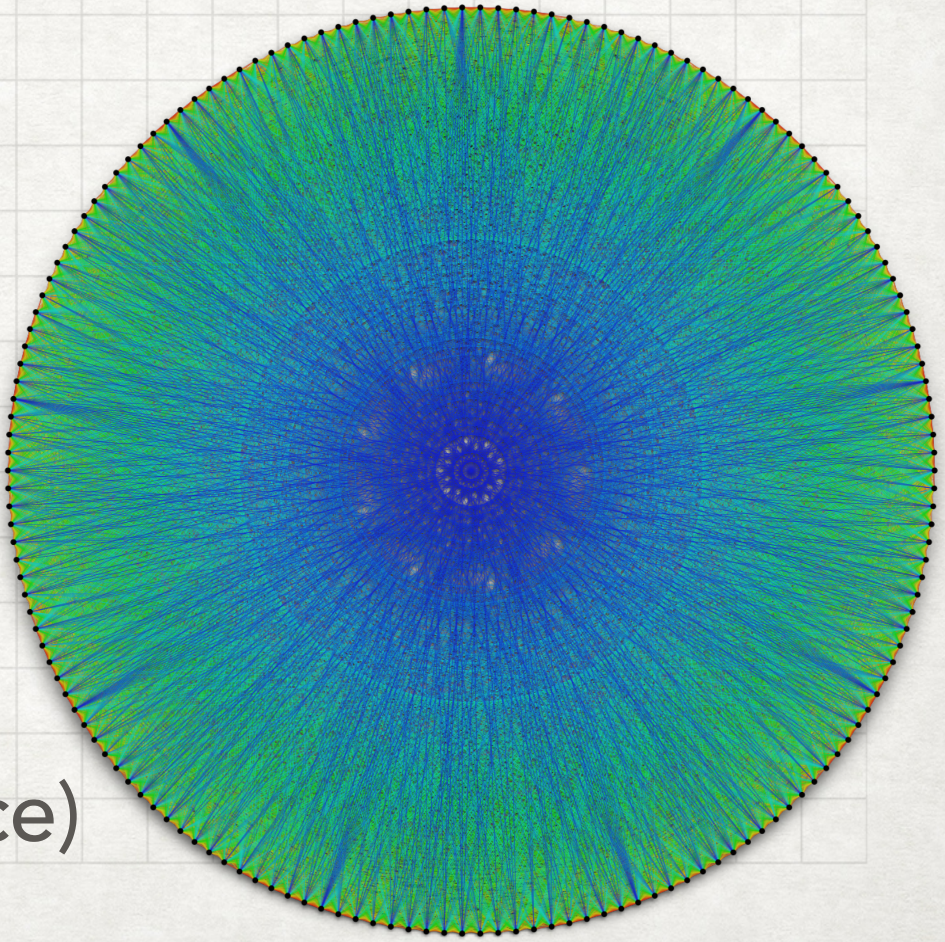


Schläfli Graph (E8 lattice)

r = 23:

276 lines

McLaughlin Graph (Leech lattice)



Theorem[Absolute bound] (Gerzon 73): $N(r) \leq \binom{r+1}{2}$.

Proof: Let $v_1, \ldots, v_n$ be unit vectors along the given lines. Then $\langle v_i, v_j \rangle = \pm\alpha$ for some $0 \leq \alpha < 1$.

Consider the matrices $v_1 v_1^\intercal, \ldots, v_n v_n^\intercal$. They live in the $\binom{r+1}{2}$-dimensional space of symmetric matrices $\mathscr{S}_r$.

Recalling the Frobenius inner product of matrices

$$\langle A, B \rangle_F = \operatorname{tr}(A^\intercal B) = \sum_{i,j} A_{i,j} B_{i,j}$$

we have $\left\langle v_i v_i^\intercal, v_j v_j^\intercal \right\rangle_F = \operatorname{tr}(v_i v_i^\intercal v_j v_j^\intercal) = (v_i^\intercal v_j)^2 = \begin{cases} 1 & i = j \\ \alpha^2 & i \neq j \end{cases}$.

Hence they are linearly independent. $\square$

What is known?

Theorem[Absolute bound] (Gerzon 73): $N(r) \leq \binom{r+1}{2}$.

- tight in dimension 2,3, 7 and 23. No other cases of equality are known.

Theorem (de Caen 00): $N(r) \geq \Omega(r^2)$.

Question (Lemmens, Seidel 73):
Determine $N_\alpha(r)$, the maximum number of equiangular lines in $\mathbb{R}^r$ with common angle $\arccos(\alpha)$, especially when $\alpha = 1/3, 1/5, 1/7, \ldots$

Theorem (Neumann 73): If $N_\alpha(r) > 2r$ then $\frac{1}{2}\left(\frac{1}{\alpha} - 1\right) \in \mathbb{N}$.

Theorem[Relative Bound] (Lemmens, Seidel 73): $N_\alpha(r) \leq r\frac{1-\alpha^2}{1-r\alpha^2}$ for all $r \leq 1/\alpha^2 - 2$.

Recent progress

Theorem (B., Dräxler, Keevash, Sudakov 17): $N_\alpha(r) \leq 2r-2$ if r is exponentially large in $1/\alpha^2$, with equality if and only if $\alpha = 1/3$.

Theorem (Jiang, Tidor, Yao, Zhang, Zhao 19): Let k_α be the minimum number of vertices in a graph with spectral radius $\frac{1}{2}\left(\frac{1}{\alpha}-1\right)$. If r is doubly exponentially large in k_α/α, then

$$N_\alpha(r) = \left\lfloor \frac{r-1}{1-1/k_\alpha} \right\rfloor.$$

Question: What about for $1/\alpha^2 - 2 \leq r \leq O(2^{1/\alpha^2})$?

Theorem (Yu 17): $N_\alpha(r) \leq \binom{1/\alpha^2-1}{2}$ for $1/\alpha^2 - 2 \leq r \leq 3/\alpha^2 - 16$.

Theorem (Glazyrin, Yu 18): $N_\alpha(r) \leq \left(\frac{2}{3\alpha^2} + \frac{4}{7}\right) r + 2$ for all $\alpha \leq \frac{1}{3}$.

New results

Theorem(B.): There exists a constant $C > 0$ such that as $\alpha \to 0$,

$$N_\alpha(r) \le \begin{cases} \binom{1/\alpha^2-1}{2} & \text{if } \frac{1}{\alpha^2} - 2 < r \le \frac{1-o(1)}{4\alpha^4} \\ (2+o(1))r & \text{if } \frac{1-o(1)}{4\alpha^4} < r \le O\left(\frac{1}{\alpha^5}\right) \\ \left(1 + \frac{1+o(1)}{4\cos^2\left(\frac{\pi}{q+2}\right)}\right) r & \text{if } \frac{1}{\alpha^{2q+1}} \ll r \le O\left(\frac{1}{\alpha^{2q+3}}\right) \text{ for integer } q \ge 2 \\ \left(\frac{5}{4} + o(1)\right) r & \text{if } 1/\alpha^{\omega(1)} \le r < 2^{1/\alpha^{4C}} \\ \left(1 + \frac{C\log(1/\alpha)}{\log\log r}\right) r & \text{if } 2^{1/\alpha^{4C}} \le r < 2^{1/\alpha^{C(k_\alpha - 1)}} \\ \left\lfloor \frac{r-1}{1-1/k_\alpha} \right\rfloor & \text{if } 2^{1/\alpha^{C(k_\alpha-1)}} \le r. \end{cases}$$

equality iff the absolute bound is met in dimension $1/\alpha^2 - 2$

always equality!

Simple lower bounds: $N_\alpha(r) \ge r$ for all α, r, and if $k_\alpha < \infty$, then $N_\alpha(r) \ge \left\lfloor \frac{r-1}{1-1/k_\alpha} \right\rfloor$.

New results for regular graphs

Corollary(B.): Let G be a k-regular graph with second and last eigenvalue λ_2, λ_n. If the spectral gap satisfies $k - \lambda_2 \ll n$, then

$$\lambda_2 \geq (1 - o(1))k^{1/3} \quad \text{and} \quad \lambda_2 \geq (1 - o(1))\sqrt{-\lambda_n}.$$

Theorem(B.): If G is a k-regular graph with $k - \lambda_2 < \frac{n}{2}$, then

$$2\left(k - \frac{(k-\lambda_2)^2}{n}\right) \leq \frac{\lambda_2(\lambda_2+1)(2\lambda_2+1)}{1 - \frac{2(k-\lambda_2)}{n}} - \lambda_2(3\lambda_2 + 1),$$

$$-\lambda_n \leq \frac{\lambda_2(\lambda_2+1)}{1 - \frac{2(k-\lambda_2)}{n}} - \lambda_2,$$

with equality in both whenever $n + 1 = \binom{n - \mathrm{mult}(\lambda_2) + 1}{2}$, i.e. when G corresponds to a set of real equiangular lines meeting the absolute bound in dimension $r = n - \mathrm{mult}(\lambda_2)$.

Corollary(B.): Let G be a k-regular graph with second eigenvalue λ_2. If the spectral gap satisfies $k - \lambda_2 \ll n$, then

$$\lambda_2 \geq (1 - o(1))k^{1/3}\ .$$

Proof sketch: Starting with the adjacency matrix A, let $\alpha = \frac{1}{2\lambda_2+1}$ and define $M = (1-\alpha)I + \alpha J - 2\alpha A$.

Straightforward to check that M is positive semidefinite, so it is the Gram matrix of some unit vectors $v_1, \ldots, v_n$.

Note that $\overline{v} = \frac{1}{n}\sum_{i=1}^{n} v_i$ is equidistant from each v_i.

Project $X = \overline{v}v_1^\intercal + v_1\overline{v}^\intercal$ onto the span of $v_1v_1^\intercal, \ldots, v_nv_n^\intercal$ (orthogonally with respect to the Frobenius inner product).

The (Frobenius) norm of X can only decrease! $\square$

New results in the complex setting

Given a pair of complex lines $U, V \subset \mathbb{C}^r$, the quantity $|\langle u, v\rangle|$ is the same for all unit vectors $u \in U, v \in V$ and so $\arccos|\langle u, v\rangle|$ is called the **Hermitian angle** between U and V.

We define $N_\alpha^{\mathbb{C}}(r)$ to be the maximum number of complex equiangular lines in $\mathbb{C}^r$ with common Hermitian angle $\arccos(\alpha)$.

Theorem[Absolute bound] (Delsarte, Goethals, Seidel 75): $N_\alpha^{\mathbb{C}}(r) \leq r^2$.

Conjecture (Zauner 99): For each $r \in \mathbb{N}$, $\max_\alpha N_\alpha^{\mathbb{C}}(r) = r^2$ and a construction can be obtained as the orbit of a vector under the action of a Weyl-Heisenberg group.

Collections of r^2 complex equiangular lines in $\mathbb{C}^r$ are known as SICs/SIC-POVMs in quantum information theory.

New results in the complex setting

Theorem[Relative Bound] (Delsarte, Goethals, Seidel 75)**:**
$N_\alpha^{\mathbb{C}}(r) \leq r\frac{1-\alpha^2}{1-r\alpha^2}$ for all $r \leq 1/\alpha^2 - 1$.

Theorem(B.)**:** If $r \leq \frac{1-o(1)}{\alpha^3}$, then $N_\alpha^{\mathbb{C}}(r) \leq \left(\frac{1}{\alpha^2} - 1\right)^2$, with equality if and only if there exists a SIC in $1/\alpha^2 - 1$ dimensions.

Otherwise $N_\alpha^{\mathbb{C}}(r) \leq \frac{1+\alpha}{\alpha} r + O\left(\frac{1}{\alpha^3}\right)$.

Future directions for research

- Unit vectors corresponding to equiangular lines are equivalently spherical $\{\alpha, -\alpha\}$-codes. Extend methods to more general spherical L-codes.
- Prove an appropriate generalization of the usual Alon-Boppana theorem to determine $N_\alpha^{\mathbb{C}}(r)$ up to a multiplicative constant.
- Generalize to equiangular subspaces.
- Generalize to signed graphs and unitarily-signed graphs.

thank you
рахмат
danke
謝謝
ngiyabonga
teşekkür ederim
Баярлалаа
спасибо
mersi
kia ora
barka
welalin
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tapadh leat
faafetai lava
vinaka
спасибі
blagodaram
dank je
misaotra
matondo
paldies
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maururu
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gràçie
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