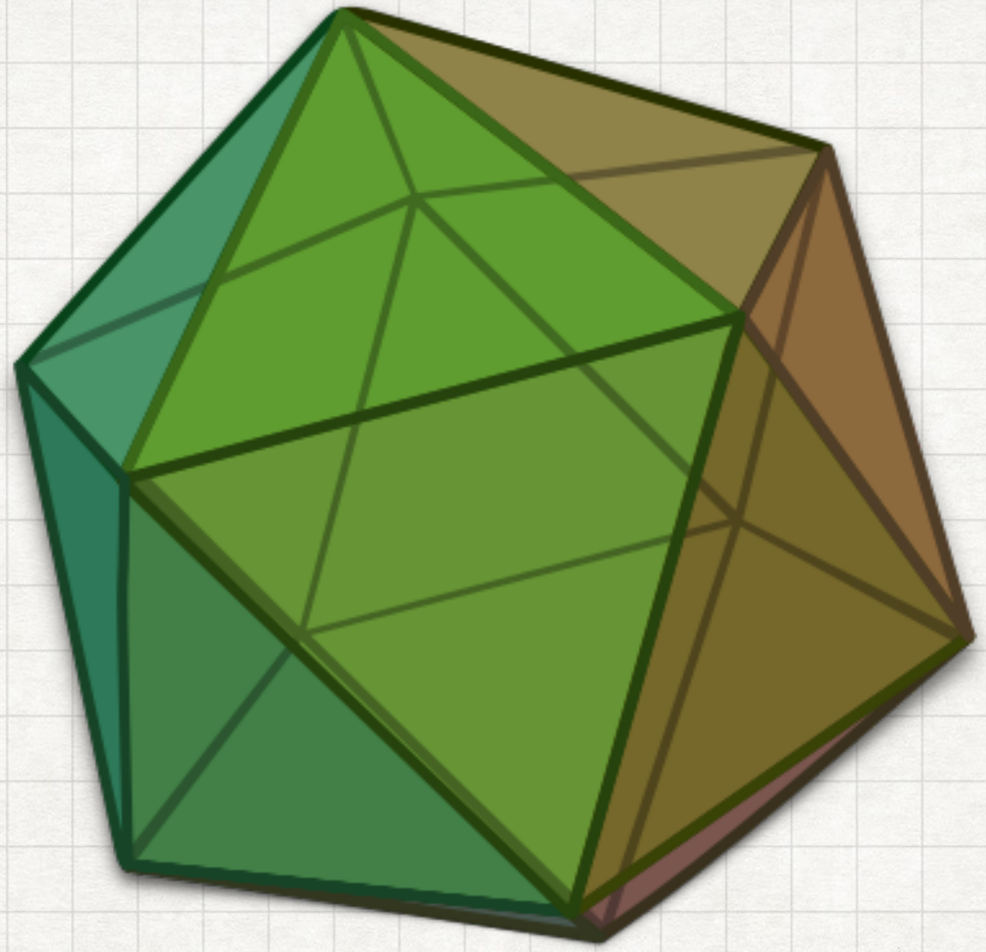 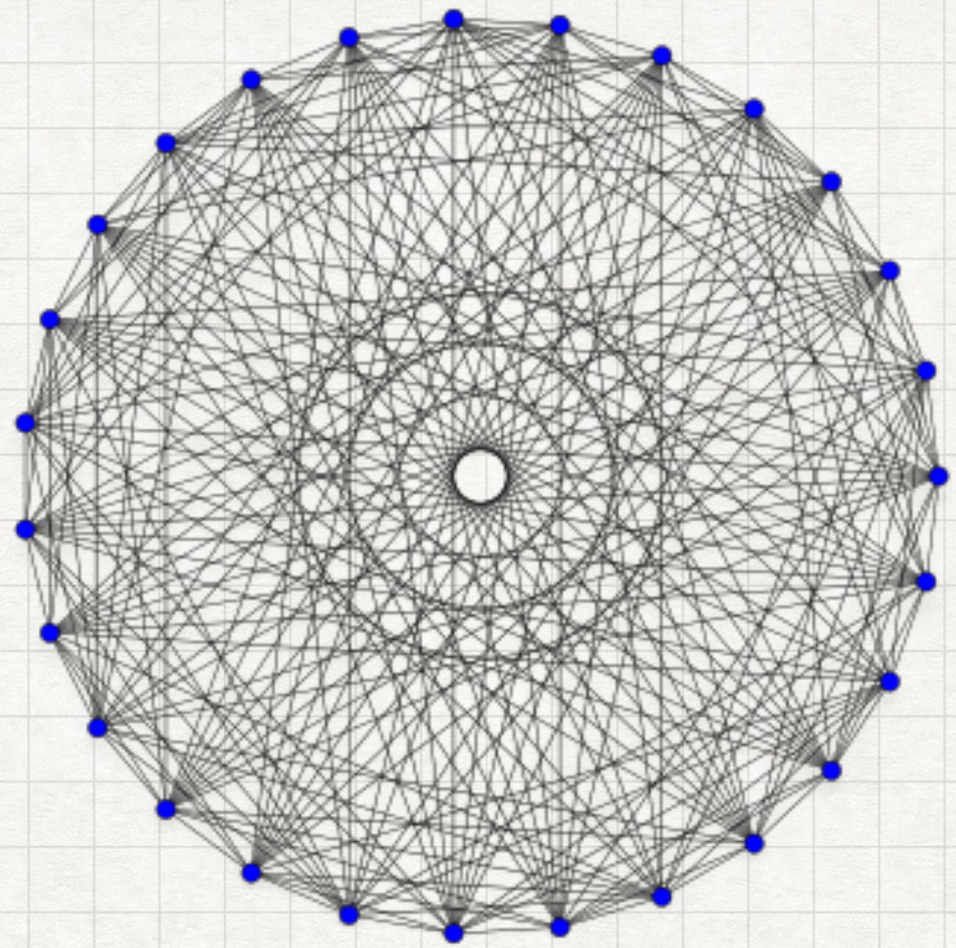

EQUIANGULAR LINES AND REGULAR GRAPHS

By: Igor Balla

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Connections:

- Elliptic geometry
- Frame theory
- Theory of polytopes
- Banach space theory
- Spectral graph theory
- Algebraic number theory
- Quantum information theory

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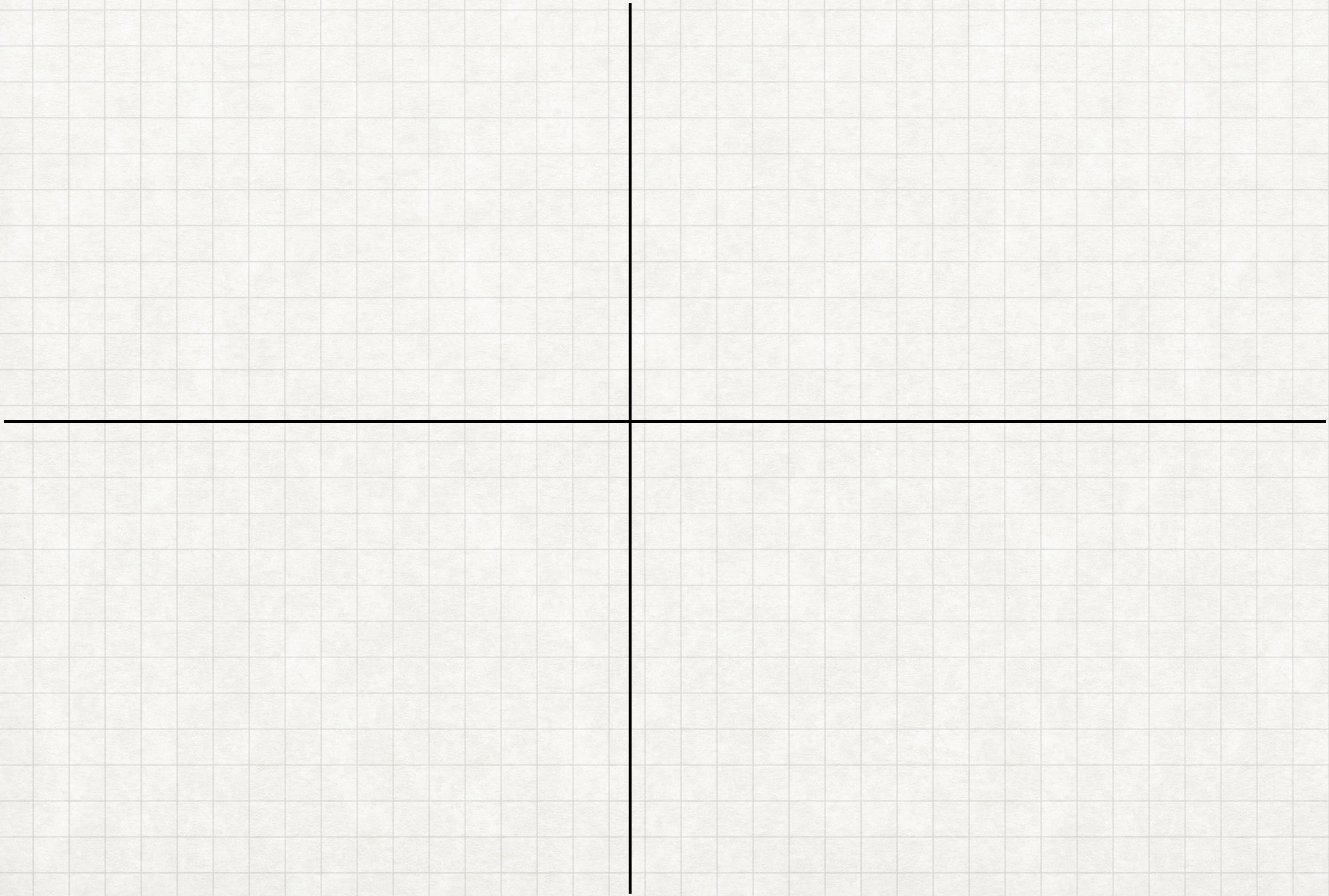
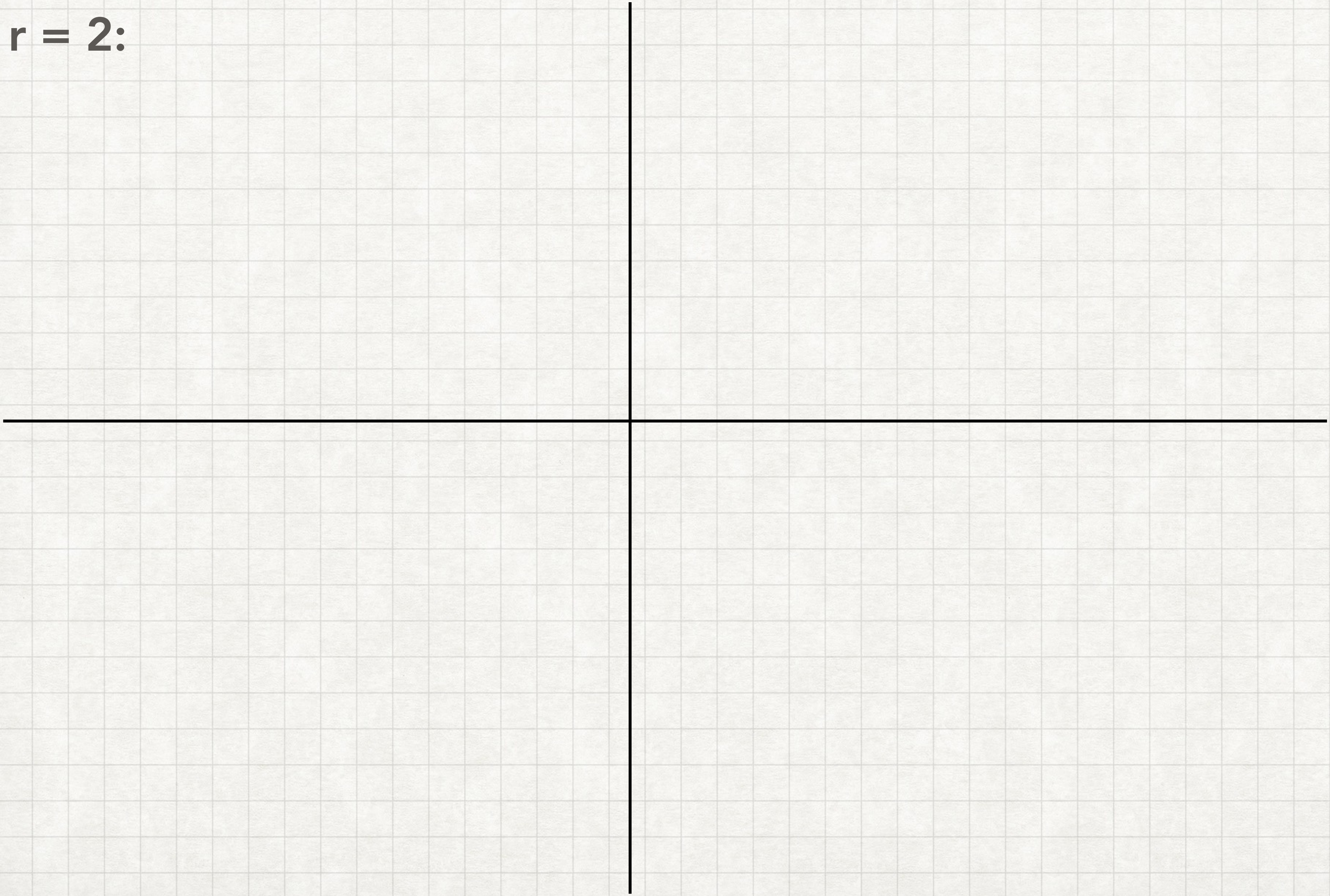
Examples



Examples

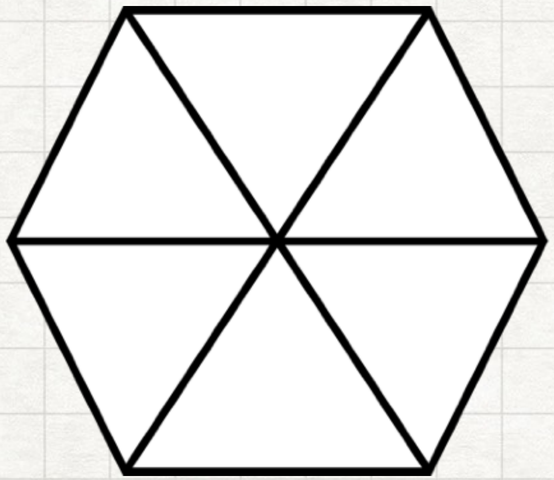
$r = 2$:



Examples

r = 2: Regular Hexagon

3 lines



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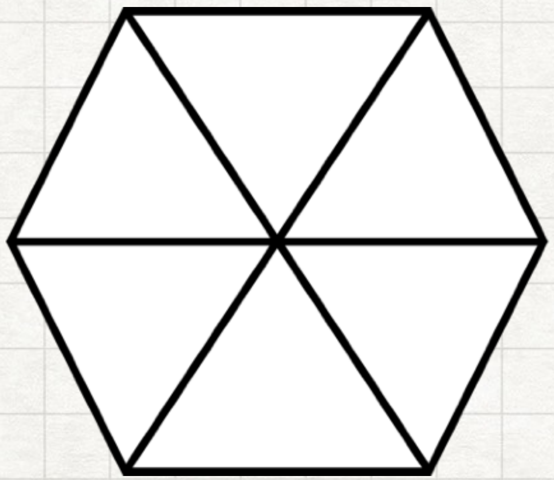

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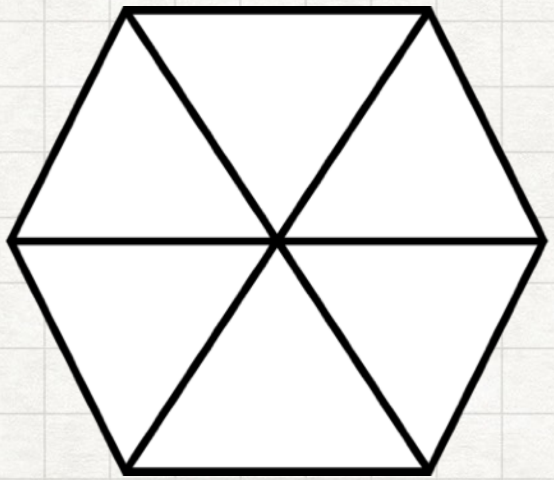
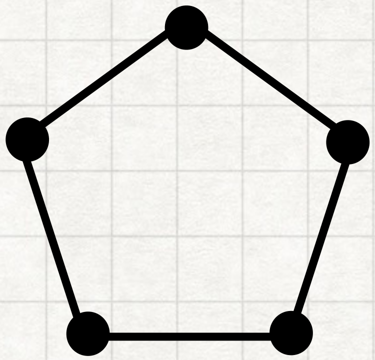
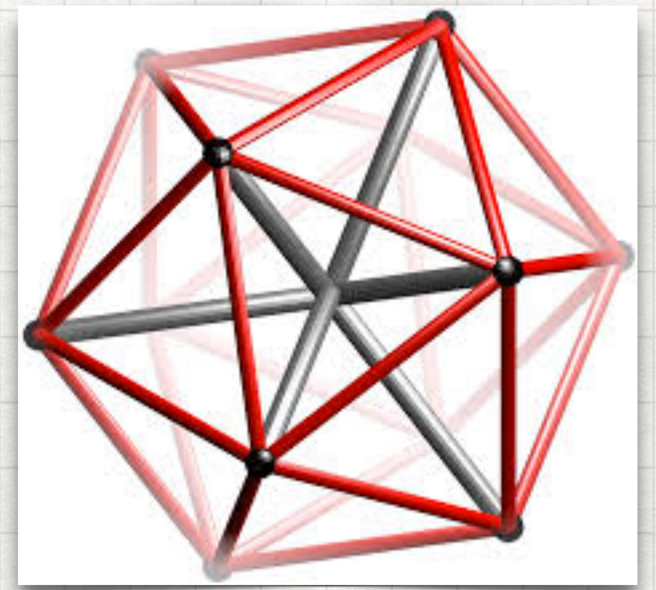


r = 3: Regular Icosahedron

6 lines





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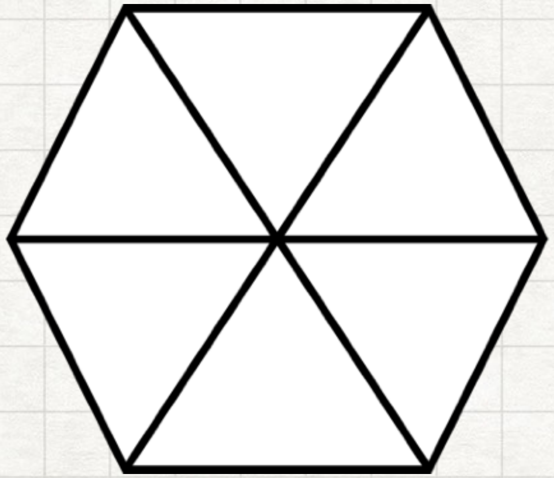

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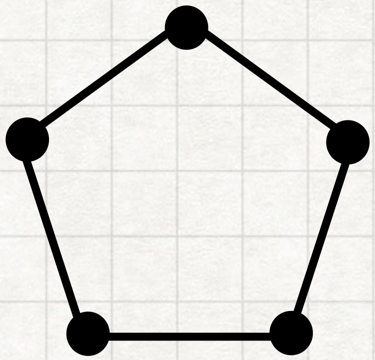

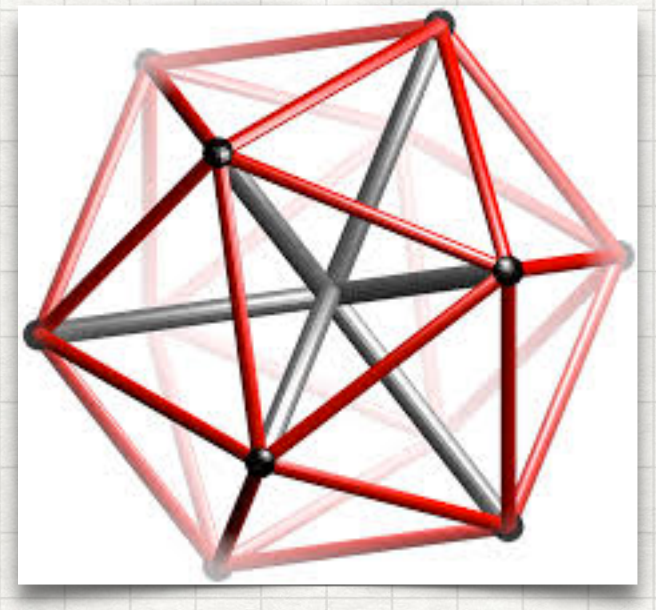

r = 7:

Examples

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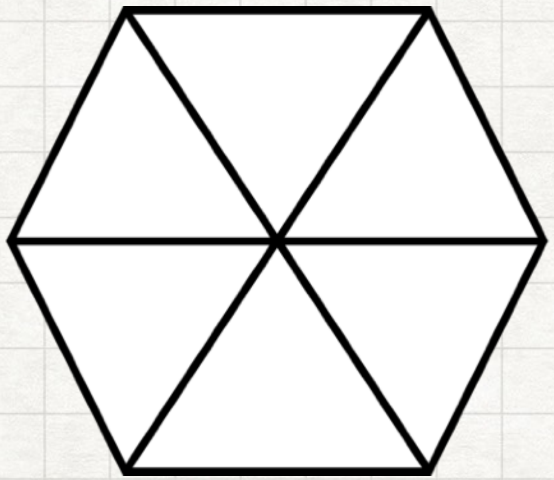

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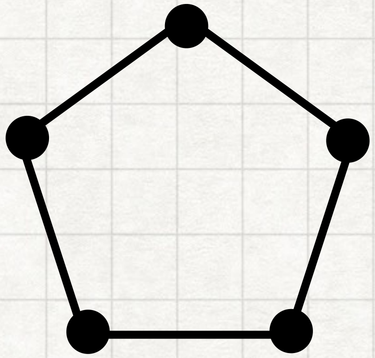
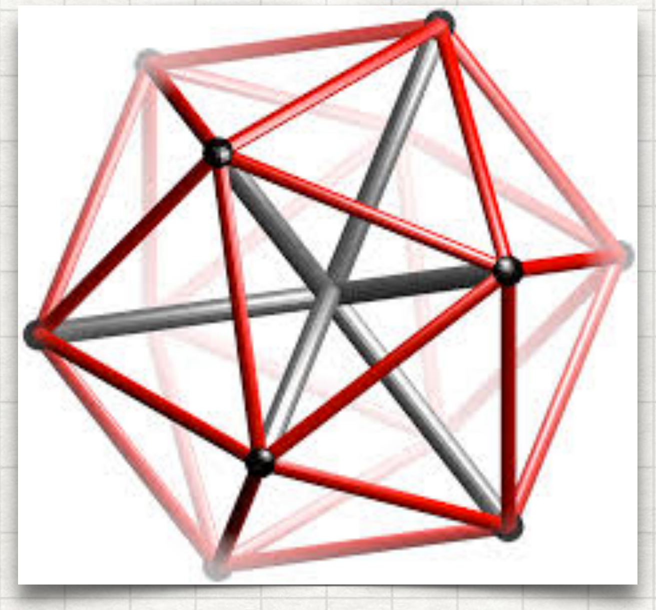
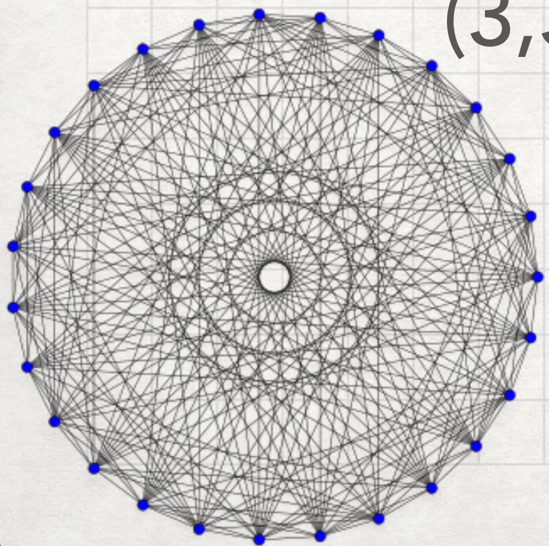




r = 7: Take all 28 permutations of the vector (3,3,-1,-1,-1,-1,-1,-1).

28 lines



Schläfli Graph (E8 lattice)

Examples

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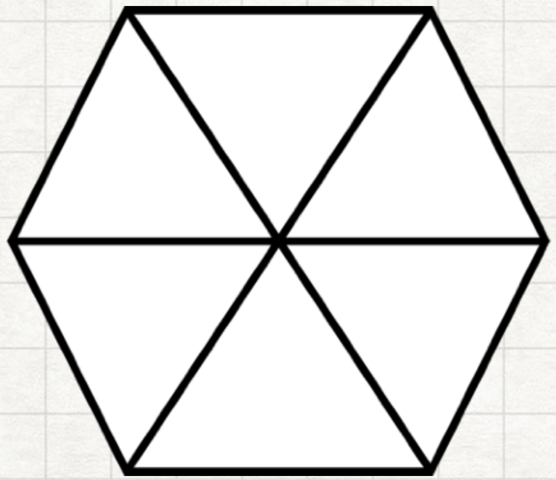

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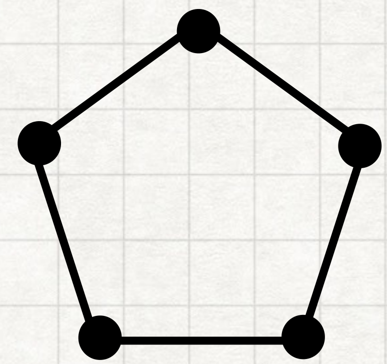

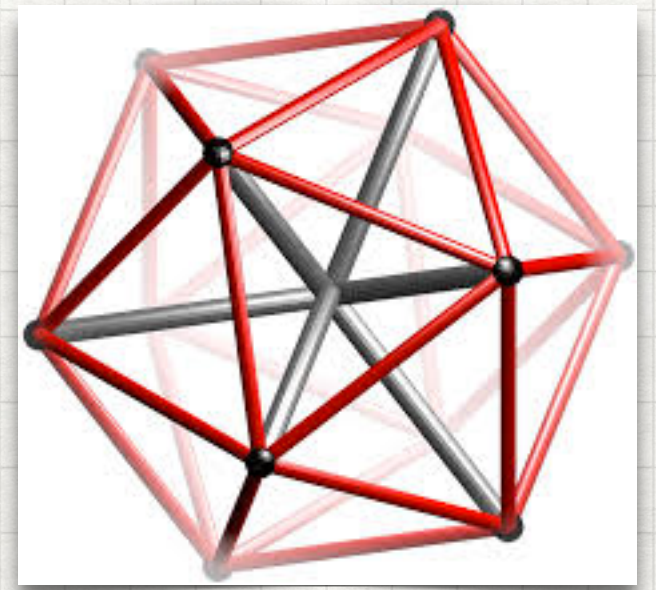

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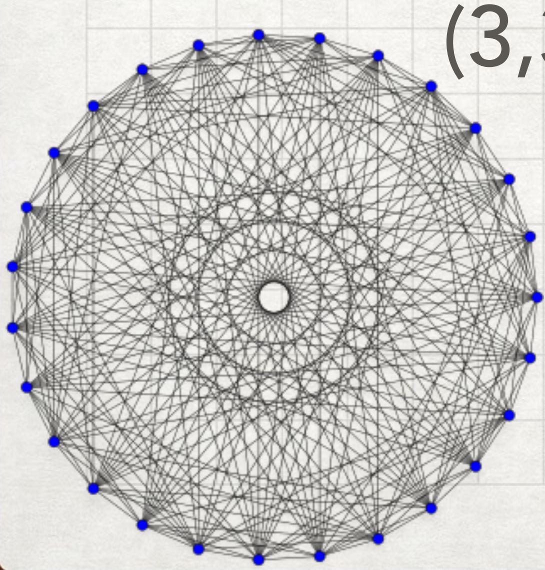

Schläfli Graph
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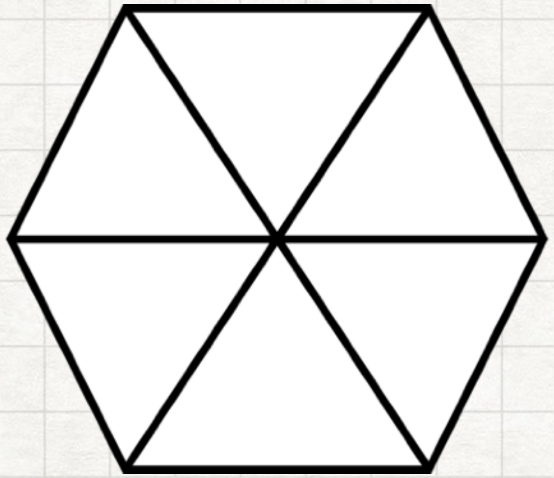
r = 23:

Examples

r = 2: Regular Hexagon

3 lines



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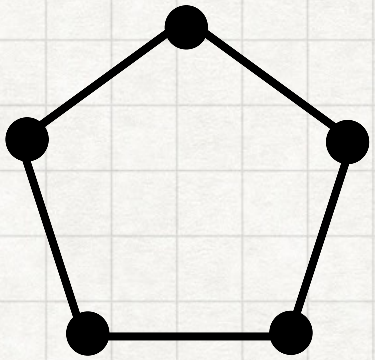

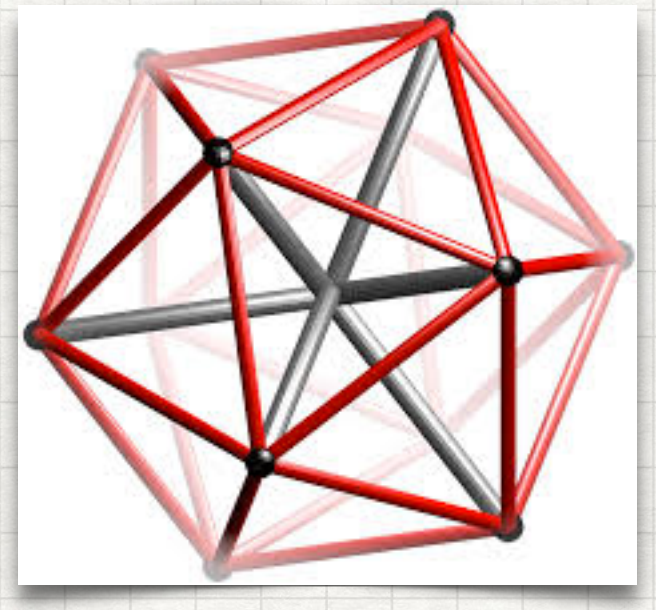

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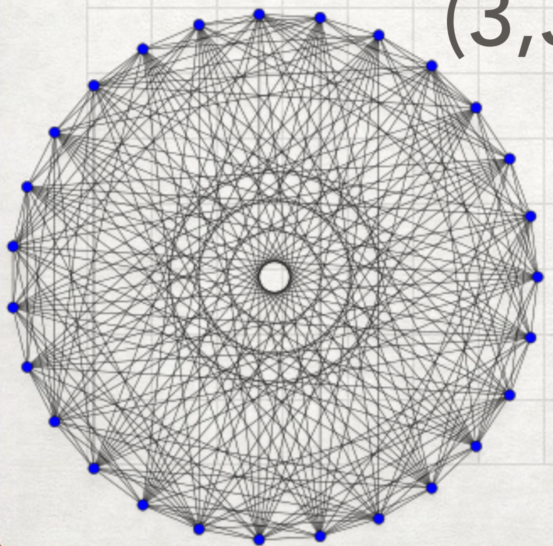
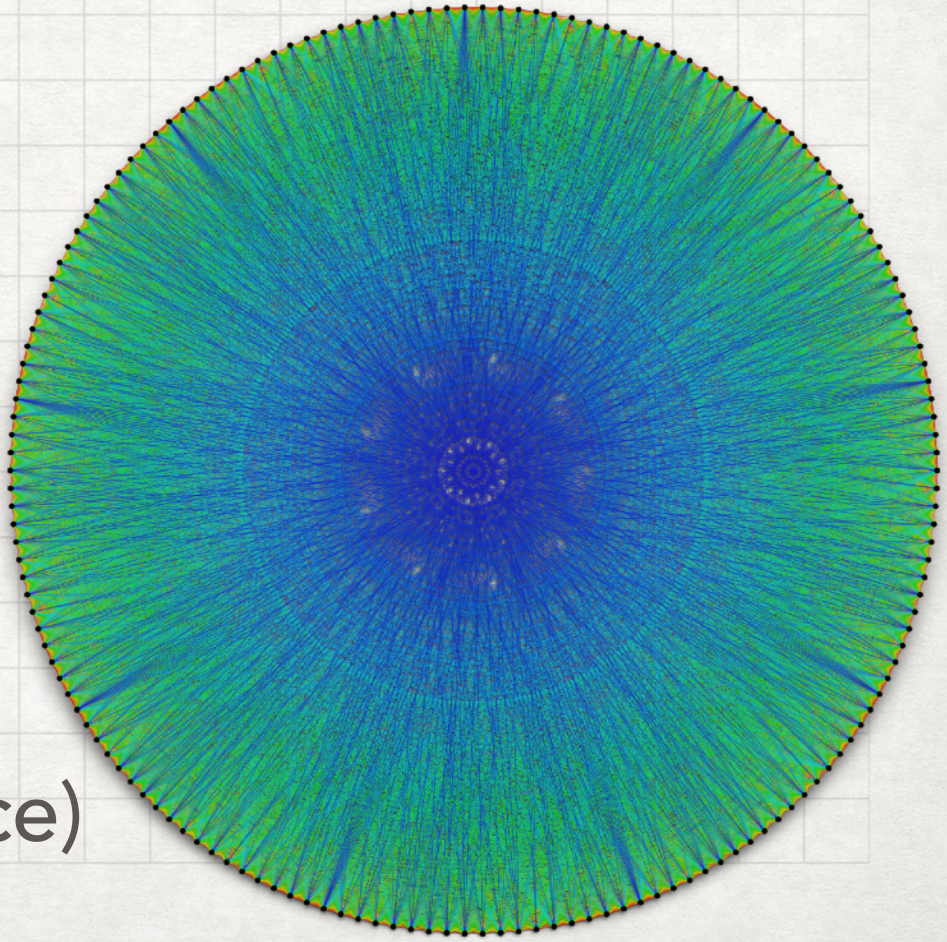


Schläfli Graph (E8 lattice)

r = 23:

276 lines

McLaughlin Graph (Leech lattice)



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Hence they are linearly independent. $\square$

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always equality when this term is bigger!

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Theorem(B.): If G is a k-regular graph with $k - \lambda_2 < \frac{n}{2}$, then

$$2\left(k - \frac{(k-\lambda_2)^2}{n}\right) \leq \frac{\lambda_2(\lambda_2+1)(2\lambda_2+1)}{1 - \frac{2(k-\lambda_2)}{n}} - \lambda_2(3\lambda_2 + 1),$$

$$-\lambda_n \leq \frac{\lambda_2(\lambda_2+1)}{1 - \frac{2(k-\lambda_2)}{n}} - \lambda_2,$$

with equality in both whenever $n + 1 = \binom{n - \mathrm{mult}(\lambda_2) + 1}{2}$, i.e. when G corresponds to a set of real equiangular lines meeting the absolute bound in dimension $r = n - \mathrm{mult}(\lambda_2)$.

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Proof sketch: Starting with the adjacency matrix A, let $\alpha = \frac{1}{2\lambda_2 + 1}$ and define $M = (1-\alpha)I + \alpha J - 2\alpha A$.

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Proof sketch: Starting with the adjacency matrix A, let $\alpha = \frac{1}{2\lambda_2+1}$ and define $M = (1-\alpha)I + \alpha J - 2\alpha A$.

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The (Frobenius) norm of X can only decrease! $\square$

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Consider the graph with vertices $v_1, \dots, v_n$ such that $v_i v_j$ forms an edge if and only if $\langle v_i, v_j \rangle = -\alpha$.

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The second bound $n \leq (2+o(1))r$ then follows by applying the inequality $\mathrm{tr}(H)^2 \leq \mathrm{rk}(H)\mathrm{tr}(H^2)$ with $H = M - \alpha J$. $\square$

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Given a pair of complex lines $U, V \subset \mathbb{C}^r$, the quantity $|\langle u, v\rangle|$ is the same for all unit vectors $u \in U, v \in V$ and so $\arccos|\langle u, v\rangle|$ is called the **Hermitian angle** between U and V.

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Collections of r^2 complex equiangular lines in $\mathbb{C}^r$ are known as SICs/SIC-POVMs in quantum information theory.

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Otherwise $N_\alpha^{\mathbb{C}}(r) \leq \frac{1+\alpha}{\alpha} r + O\left(\frac{1}{\alpha^3}\right)$.

Future directions for research

- Unit vectors corresponding to equiangular lines are equivalently spherical $\{\alpha, -\alpha\}$-codes. Extend methods to more general spherical L-codes.
- Determine $N_\alpha^{\mathbb{C}}(r)$ up to a multiplicative constant.
- Generalize to other graph matrices (ex: Laplacian).
- Generalize to equiangular subspaces.
- Generalize to signed graphs and unitarily-signed graphs.

рахмат
danke
謝謝
ngiyabonga
teşekkür ederim
spas
Баярлалаа
mersi
kia ora
barka
welalin
tack
mahalo
tapadh leat
спасибо
faafetai lava
vinaka
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dank je
misaotra
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thank you
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obrigada
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mercé
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