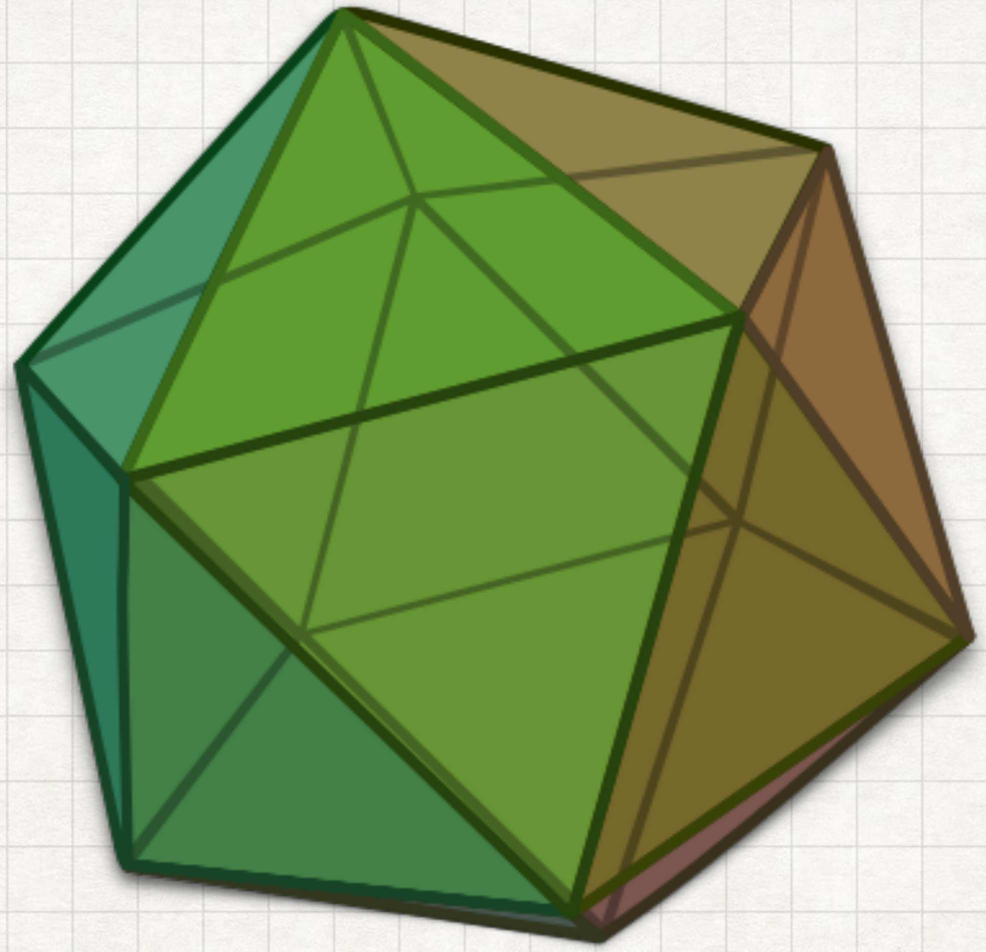 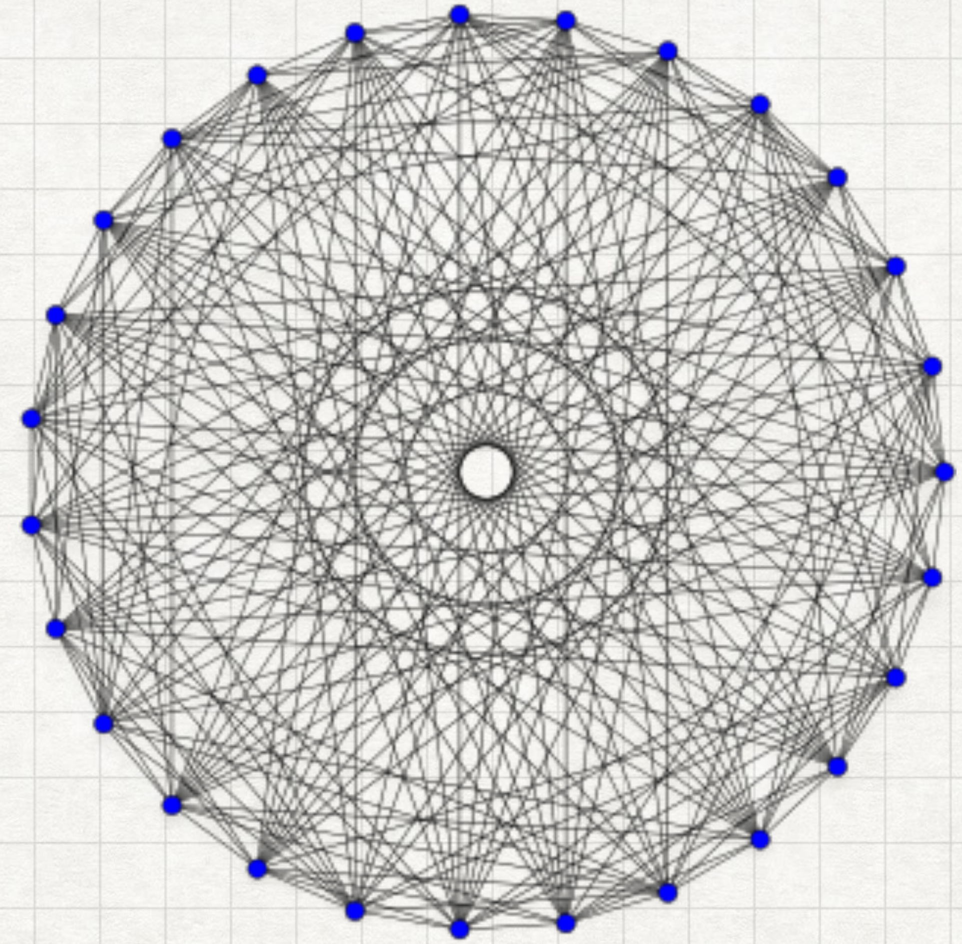

EQUIANGULAR LINES VIA MATRIX PROJECTION

By: Igor Balla

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Connections:

- Elliptic geometry
- Frame theory
- Theory of polytopes
- Banach space theory
- Spectral graph theory
- Algebraic number theory
- Quantum information theory

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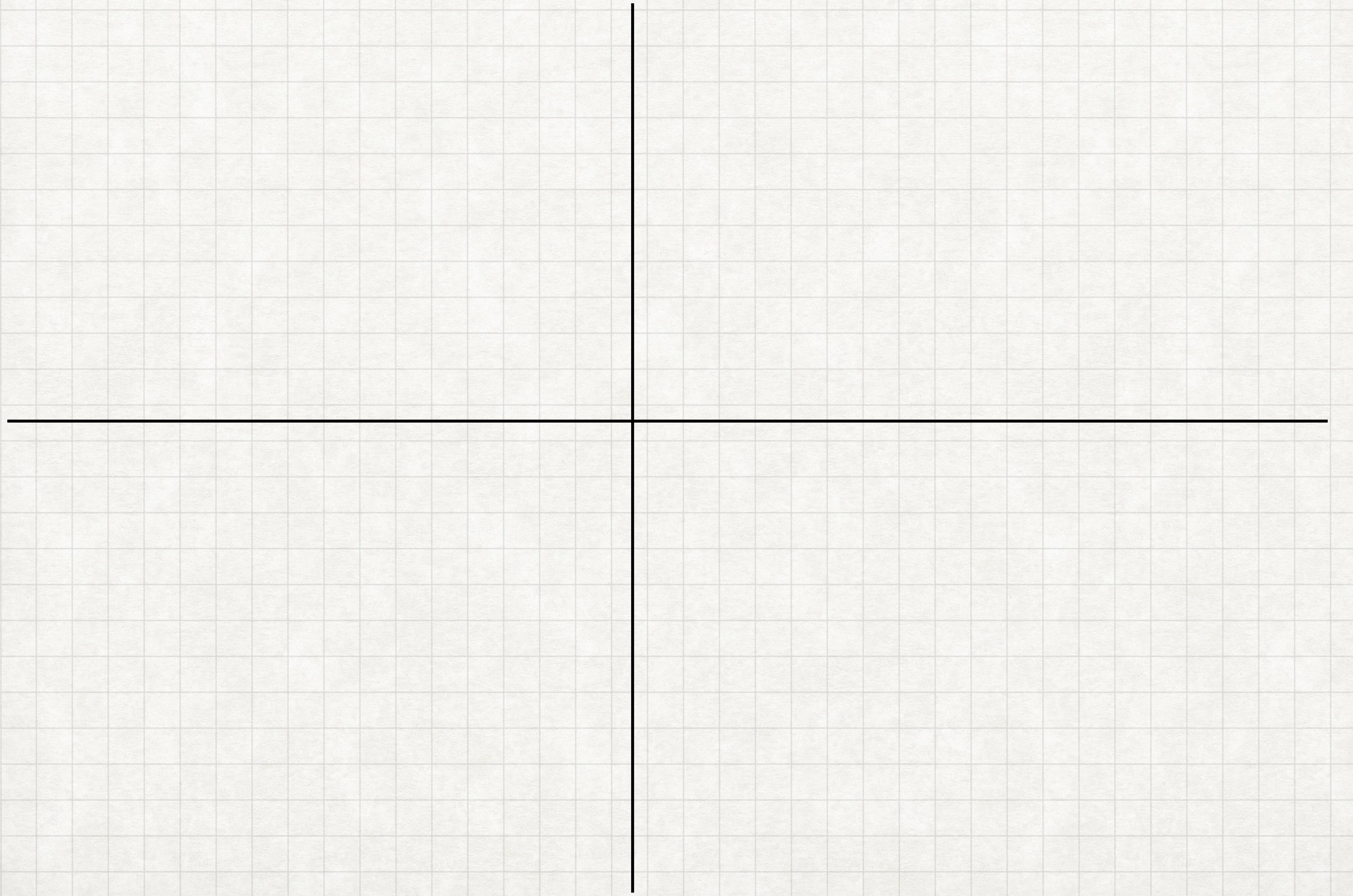
Examples



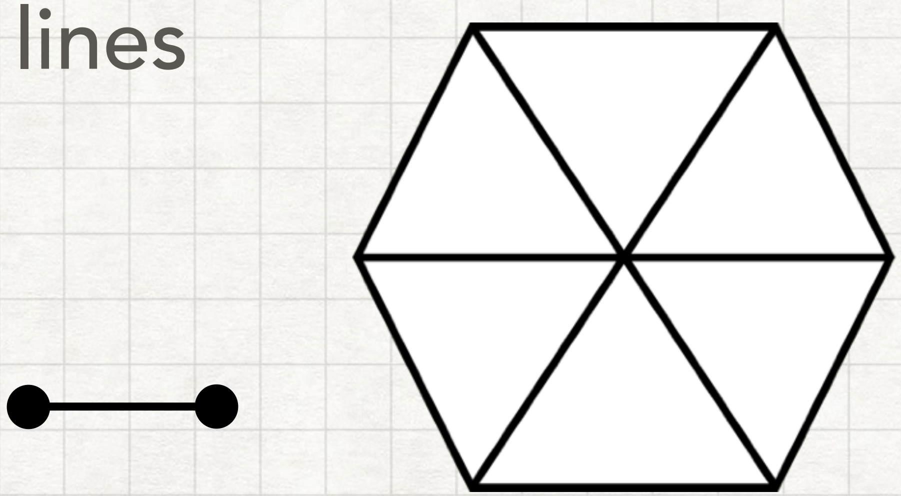
Examples

r = 2:

Examples

r = 2: Regular Hexagon

3 lines



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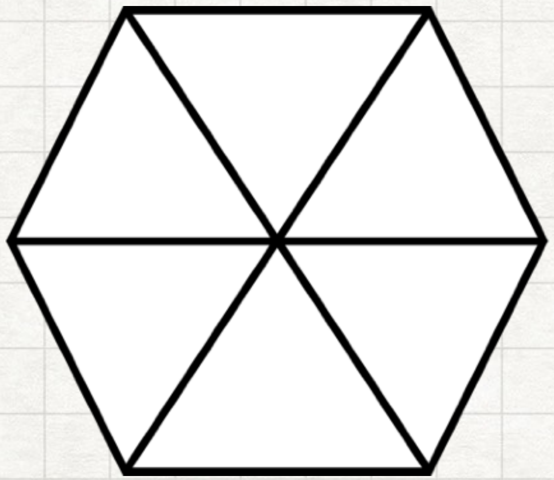

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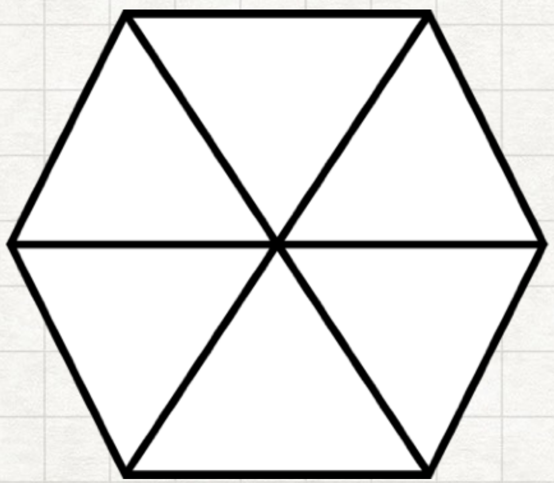
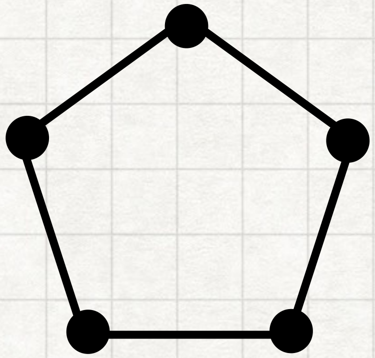
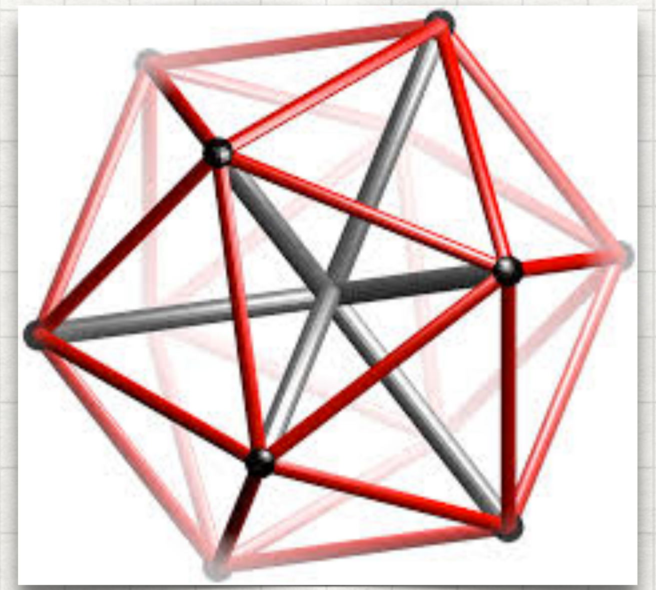


r = 3: Regular Icosahedron

6 lines





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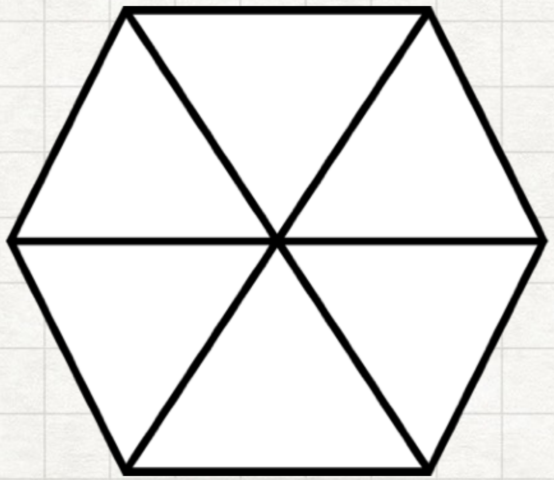

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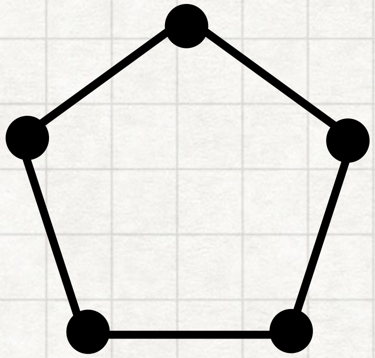

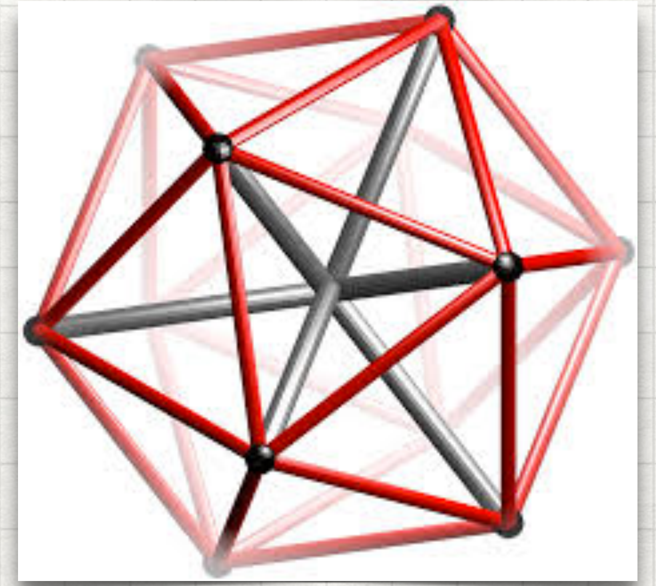

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Examples

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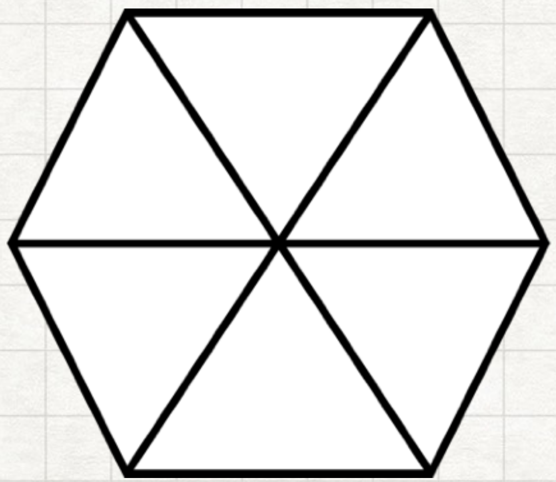

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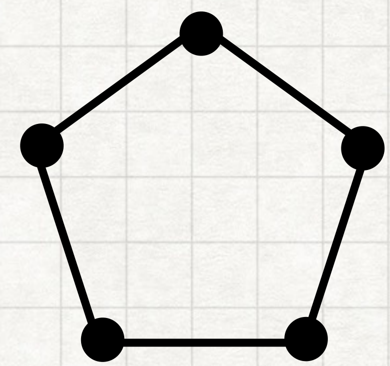

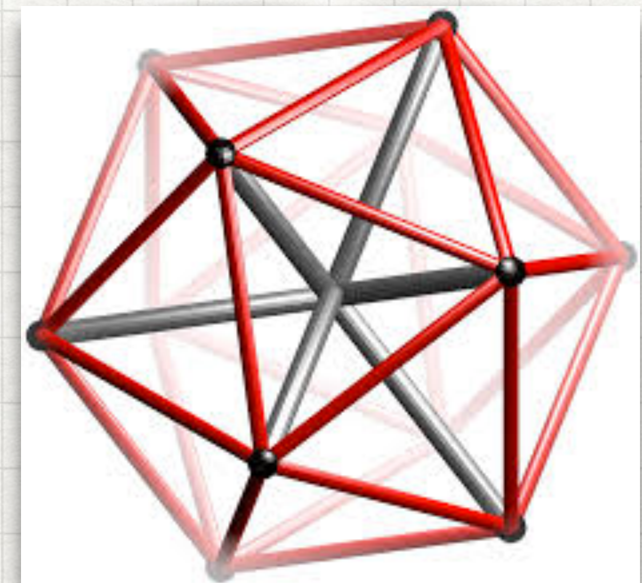

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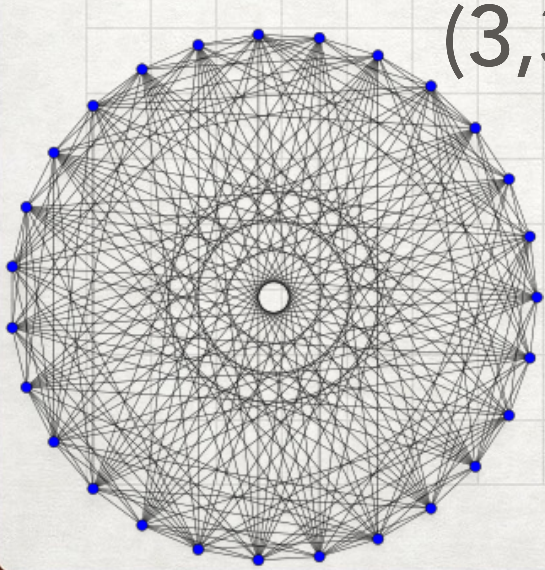
28 lines



Schläfli Graph
(E8 lattice)

Examples

r = 2: Regular Hexagon

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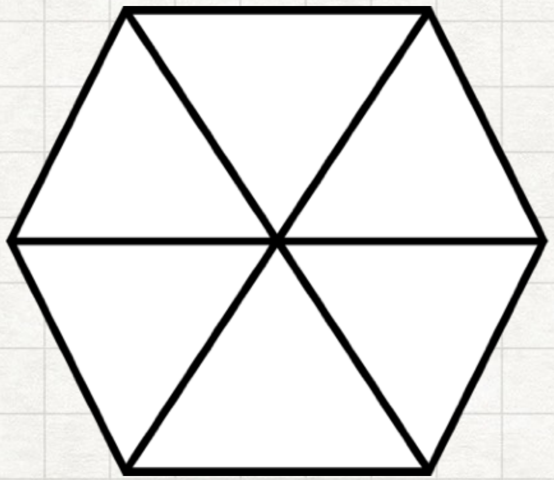

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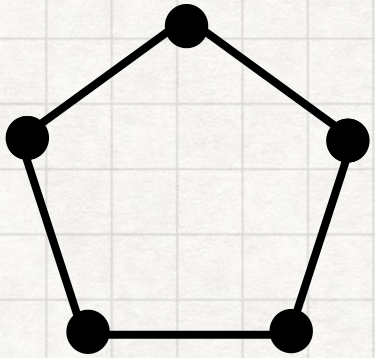

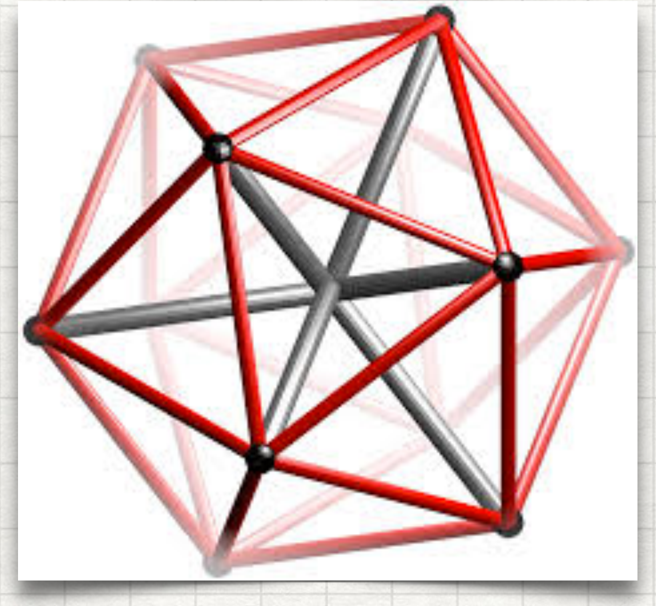

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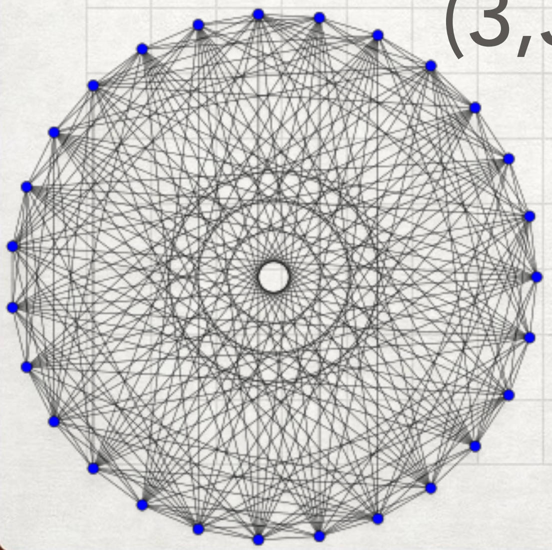


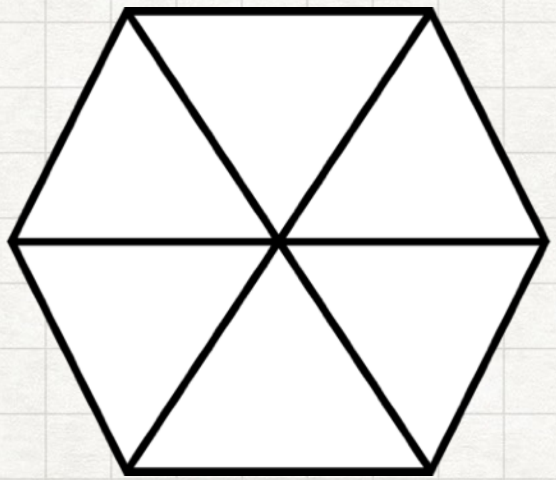
Schläfli Graph (E8 lattice)

r = 23:

Examples

r = 2: Regular Hexagon

3 lines



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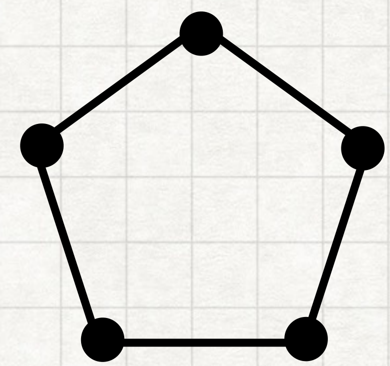

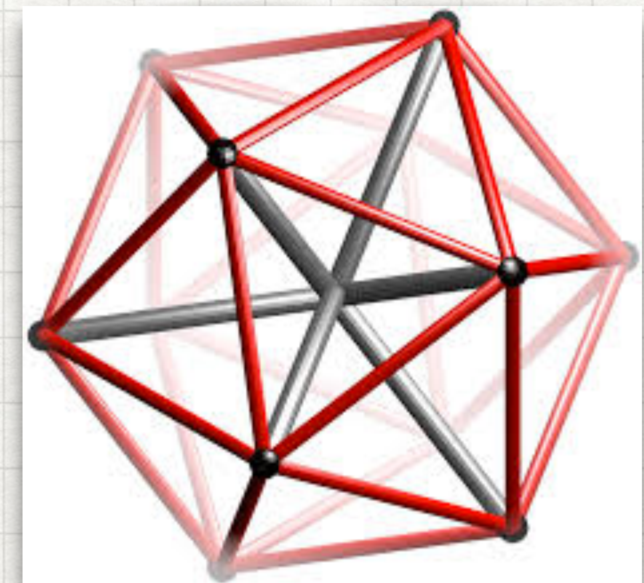

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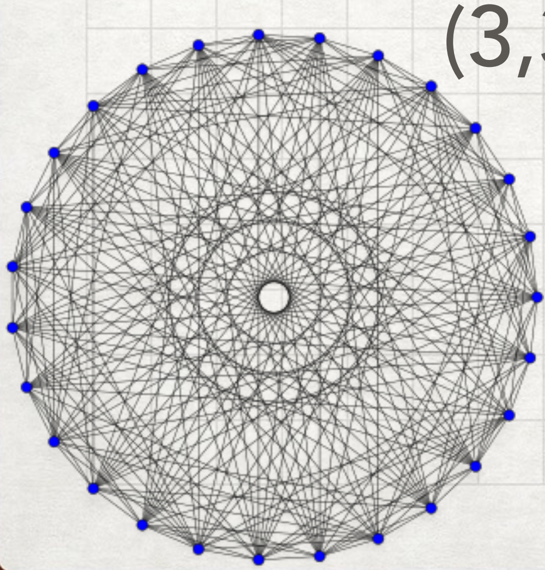

Schläfli Graph (E8 lattice)

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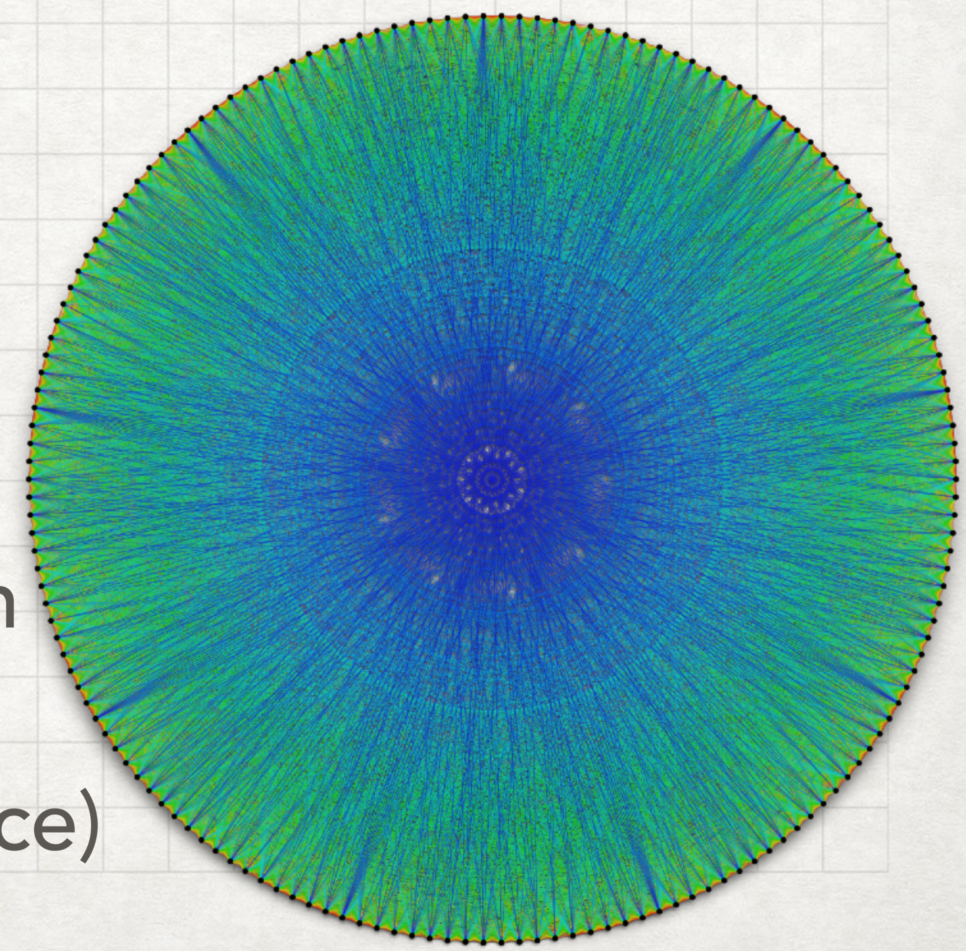
McLaughlin Graph (Leech lattice)



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Hence they are linearly independent. $\square$

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Theorem[Relative Bound] (Lemmens, Seidel 73): $N_\alpha(r) \leq r\frac{1-\alpha^2}{1-r\alpha^2}$ for all $r \leq 1/\alpha^2 - 2$.

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Theorem (Glazyrin, Yu 18): $N_\alpha(r) \le O(r/\alpha^2)$ for all $\alpha \le \frac{1}{3}$.

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Theorem (B., Bucic 24): For any positive integer k, if r is exponentially large in k^{20}, then

$$N_{\frac{1}{2k-1}}(r) = \left\lfloor \frac{r-1}{1-1/k} \right\rfloor.$$

New results for regular graphs

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Corollary(B.)**:** Let G be a k-regular graph with second and last eigenvalue λ_2, λ_n. If the spectral gap satisfies $k - \lambda_2 \ll n$, then

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Theorem(B.): If G is a k-regular graph with $k - \lambda_2 < \frac{n}{2}$, then

$$2\left(k - \frac{(k-\lambda_2)^2}{n}\right) \leq \frac{\lambda_2(\lambda_2+1)(2\lambda_2+1)}{1 - \frac{2(k-\lambda_2)}{n}} - \lambda_2(3\lambda_2 + 1),$$

$$-\lambda_n \leq \frac{\lambda_2(\lambda_2+1)}{1 - \frac{2(k-\lambda_2)}{n}} - \lambda_2,$$

with equality in both whenever $n + 1 = \binom{n - \mathrm{mult}(\lambda_2) + 1}{2}$, i.e. when G corresponds to a set of real equiangular lines meeting the absolute bound in dimension $r = n - \mathrm{mult}(\lambda_2)$.

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Orthogonally project $\left(\sum_{j=1}^n v_j\right) v_1^\intercal + v_1 \left(\sum_{j=1}^n v_j\right)^\intercal$ onto the span of $v_1 v_1^\intercal, \ldots, v_n v_n^\intercal$ (with respect to the Frobenius inner product).

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The (Frobenius) norm of X can only decrease! $\square$

Theorem(B.): Assuming $\alpha \to 0$, we have

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Consider the graph with vertices $v_1, \ldots, v_n$ such that $v_i v_j$ forms an edge if and only if $\langle v_i, v_j \rangle = -\alpha$.

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The second bound $n \le (2+o(1))r$ then follows by applying the inequality $\mathrm{tr}(H)^2 \le \mathrm{rk}(H)\mathrm{tr}(H^2)$ with $H = M - \alpha J$. □

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Definition: Let $\mathscr{W}^\# : \mathbb{R}^{r\times r} \to \mathbb{R}^n$ denote the adjoint map with respect to the Frobenius inner product ($\langle M, \mathscr{W} v\rangle_F = \langle \mathscr{W}^\# M, v\rangle$ for all matrices $M \in \mathbb{R}^{r\times r}$ and vectors $v \in \mathbb{R}^n$).

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Thus $\langle \mathscr{P}I, \mathscr{P}I\rangle_F = \frac{1}{1-\alpha^2}\mathbb{1}^\intercal\left(I - \frac{1}{n+1/\alpha^2-1}J\right)\mathbb{1} = \frac{n}{\alpha^2 n+1-\alpha^2}.$ □

New results in the complex setting

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Collections of r^2 complex equiangular lines in $\mathbb{C}^r$ are known as SIC-POVMs/SICs in quantum information theory.

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Otherwise $N_\alpha^{\mathbb{C}}(r) \leq \frac{1+\alpha}{\alpha} r + O\left(\frac{1}{\alpha^3}\right)\,.$

Future directions for research

- Unit vectors corresponding to equiangular lines are equivalently spherical $\{\alpha, -\alpha\}$-codes. Extend methods to more general spherical L-codes.
- Determine $N_\alpha^{\mathbb{C}}(r)$ up to a multiplicative constant.
- Generalize to other graph matrices (ex: Laplacian).
- Generalize to equiangular subspaces.
- Generalize to signed graphs and unitarily-signed graphs.


thank you
danke
謝謝
ngiyabonga
teşekkür ederim
рахмат
Баярлалаа
спасибо
mersi
kia ora
barka
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