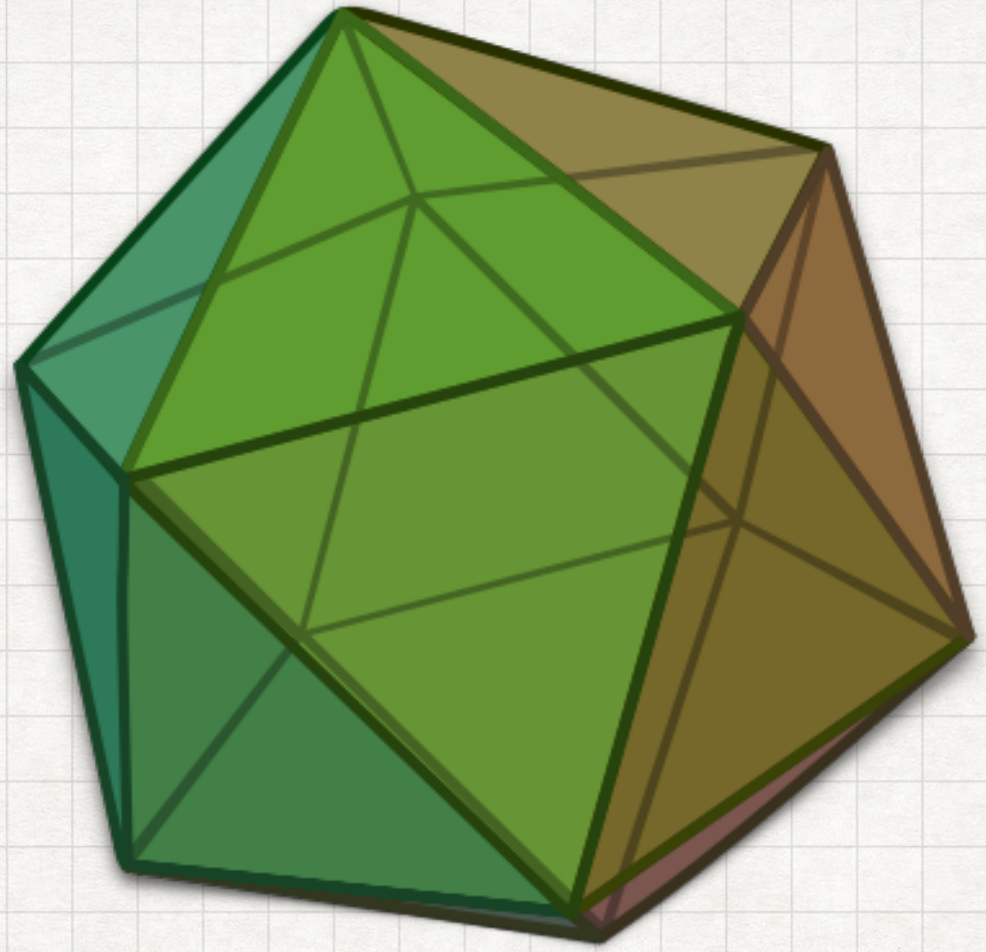
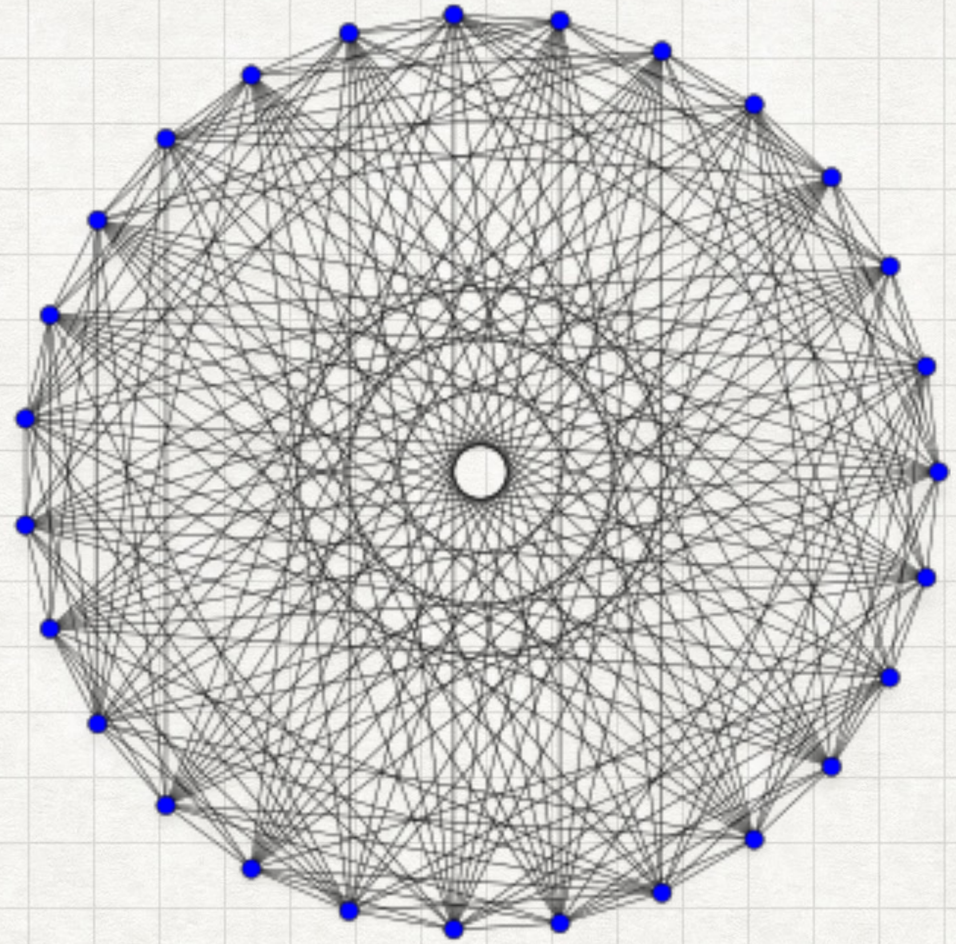




STATE OF THE ART ON EQUIANGULAR LINES

By: Igor Balla

Definition: A set of lines passing through the origin is called **equiangular**, if every pair of lines make the same angle.

Question: Determine $N(r)$, the maximum number of equiangular lines in $\mathbb{R}^r$.

Connections:

- Elliptic geometry
- Frame theory
- Theory of polytopes
- Banach space theory
- Spectral graph theory
- Algebraic number theory
- Quantum information theory

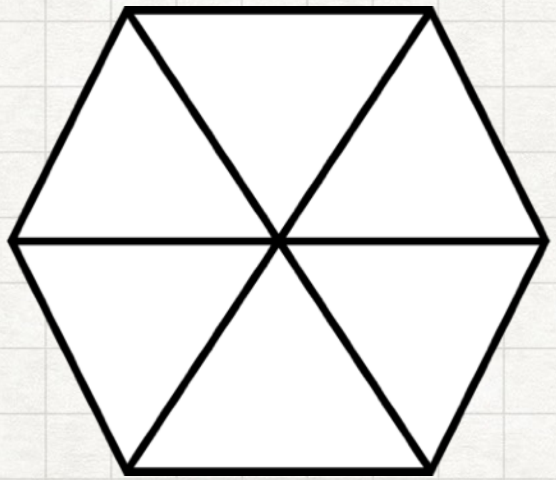
Earliest work:

Haantjes, Seidel 47-48
Blumenthal 49
Van Lint, Seidel 66
Lemmens, Seidel 73
…

Examples

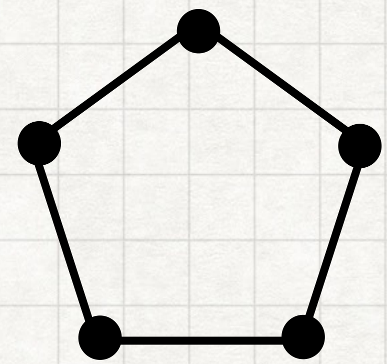
r = 2: Regular Hexagon

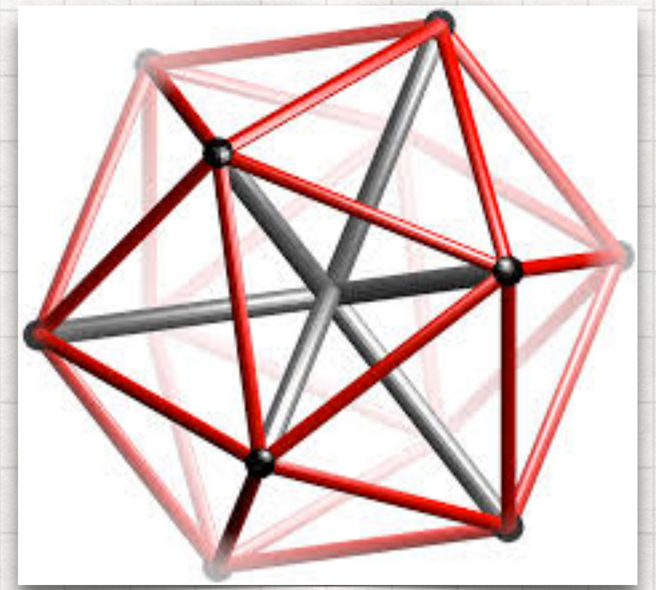
3 lines



r = 3: Regular Icosahedron

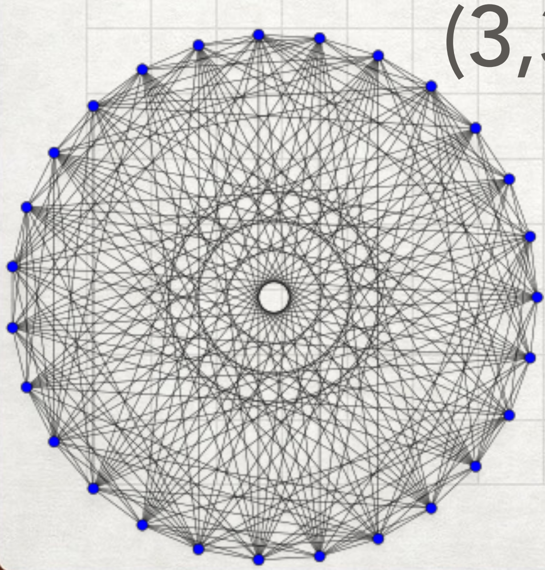
6 lines





r = 7: Take all 28 permutations of the vector (3,3,-1,-1,-1,-1,-1,-1).

28 lines

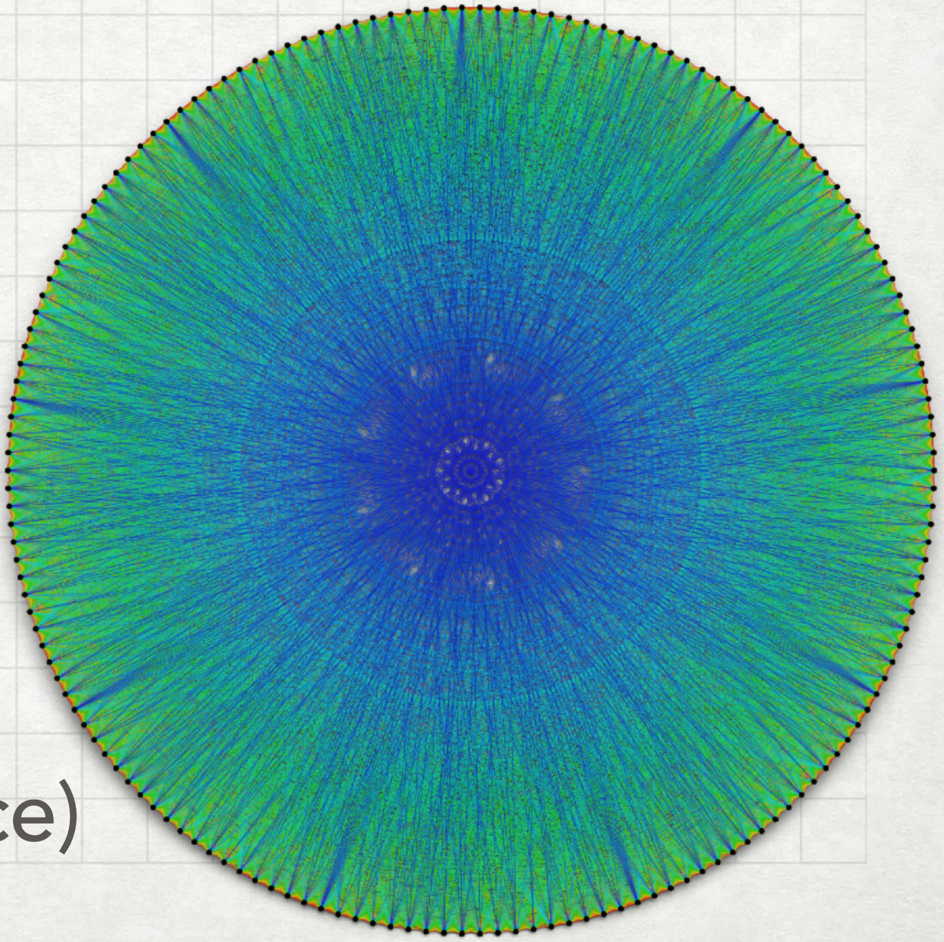


Schläfli Graph (E8 lattice)

r = 23:

276 lines

McLaughlin Graph (Leech lattice)



Theorem[Absolute bound] (Gerzon 73): $N(r) \le \binom{r+1}{2}$.

Proof: Let $v_1, \ldots, v_n$ be unit vectors along the given lines. Then $\langle v_i, v_j \rangle = \pm\alpha$ for some $0 \le \alpha < 1$.

Consider the matrices $v_1 v_1^\intercal, \ldots, v_n v_n^\intercal$. They live in the space of symmetric $r \times r$ matrices, which has dimension $\binom{r+1}{2}$.

Recalling the Frobenius inner product of matrices

$$\langle A, B \rangle_F = \mathrm{tr}(A^\intercal B) = \sum_{i,j} A_{i,j} B_{i,j}$$

we have $\left\langle v_i v_i^\intercal, v_j v_j^\intercal \right\rangle_F = \mathrm{tr}(v_i v_i^\intercal v_j v_j^\intercal) = (v_i^\intercal v_j)^2 = \begin{cases} 1 & i = j \\ \alpha^2 & i \neq j \end{cases}.$

Hence they are linearly independent. $\square$

What is known?

Theorem[Absolute bound] (Gerzon 73): $N(r) \leq \binom{r+1}{2}$.

- tight in dimension 2,3, 7 and 23. No other cases of equality are known.

Theorem (de Caen 00): $N(r) \geq \Omega(r^2)$.

Question (Lemmens, Seidel 73):

Determine $N_\alpha(r)$, the maximum number of equiangular lines in $\mathbb{R}^r$ with common angle $\arccos(\alpha)$, especially when $\alpha = 1/3, 1/5, 1/7, \ldots$

Theorem (Neumann 73): If $N_\alpha(r) > 2r$ then $\frac{1}{2}\left(\frac{1}{\alpha} - 1\right) \in \mathbb{N}$.

Fact: $N_\alpha(r) \geq r$.

Theorem[Relative Bound] (Lemmens, Seidel 73): $N_\alpha(r) \leq r\frac{1-\alpha^2}{1-r\alpha^2}$ for all $r \leq 1/\alpha^2 - 2$.

Recent progress

Theorem (B., Dräxler, Keevash, Sudakov 17): $N_\alpha(r) \le 2r-2$ if r is exponentially large in $1/\alpha^2$, with equality if and only if $\alpha = 1/3$.

Theorem (Jiang, Tidor, Yao, Zhang, Zhao 19): Let k_α be the minimum number of vertices in a graph with spectral radius $\frac{1}{2}\left(\frac{1}{\alpha}-1\right)$. If r is doubly exponentially large in k_α/α, then

$$N_\alpha(r) = \left\lfloor \frac{r-1}{1-1/k_\alpha} \right\rfloor.$$

Question: What about for $1/\alpha^2 - 2 \le r \le O(2^{1/\alpha^2})$?

Theorem (Yu 17): $N_\alpha(r) \le \binom{1/\alpha^2-1}{2}$ for $1/\alpha^2 - 2 \le r \le 3/\alpha^2 - 16$.

Theorem (Glazyrin, Yu 18): $N_\alpha(r) \le O(r/\alpha^2)$ for all $\alpha \le \frac{1}{3}$.

New results

Theorem(B.): $N_\alpha(r) \le \max\left(\binom{1/\alpha^2-1}{2}, 2r-2\right)$ for $\alpha \le 1/3$.

always equality when this term is bigger!

Conjecture(B.): $N_\alpha(r) \le \max\left(\binom{1/\alpha^2-1}{2}, \left\lfloor \frac{r-1}{1-1/k_\alpha} \right\rfloor\right)$.

- verified for $\alpha = 1/3$ by Lemmens and Seidel in 1973 and for $\alpha = 1/5$ by Cao, Koolen, Lin, and Yu in 2022 (building on the work of Neumaier)

Theorem(B., Bucić 24): If $\alpha \to 0$ and $r \ge 1/\alpha^{\omega(1)}$, then

$$N_\alpha(r) = (1+o(1))r.$$

Theorem(B., Bucić 24): For any positive integer k, if r is exponentially large in k^{20}, then

$$N_{\frac{1}{2k-1}}(r) = \left\lfloor \frac{r-1}{1-1/k} \right\rfloor.$$

New bounds for graph eigenvalues
(of adjacency matrix)

Corollary(B.): Let G be a k-regular graph on n vertices with second and last eigenvalue λ_2, λ_n. If $k - \lambda_2 \ll n$, then

$$\lambda_2 \geq (1 - o(1))k^{1/3} \quad \text{and} \quad \lambda_2 \geq (1 - o(1))\sqrt{-\lambda_n}.$$

Theorem(B.): If $k - \lambda_2 < \frac{n}{2}$, then

$$2\left(k - \frac{(k-\lambda_2)^2}{n}\right) \leq \frac{\lambda_2(\lambda_2+1)(2\lambda_2+1)}{1-\frac{2(k-\lambda_2)}{n}} - \lambda_2(3\lambda_2 + 1),$$

$$-\lambda_n \leq \frac{\lambda_2(\lambda_2+1)}{1-\frac{2(k-\lambda_2)}{n}} - \lambda_2,$$

with equality in both whenever $n + 1 = \binom{n-\mathrm{mult}(\lambda_2)+1}{2}$, i.e. when G corresponds to a set of real equiangular lines meeting the absolute bound in dimension $r = n - \mathrm{mult}(\lambda_2)$.

New bounds for graph eigenvalues

Theorem(B.): Let G be a k-regular graph on n vertices with second eigenvalue λ_2 and let $\varepsilon > 0$ be fixed. Then

$$\lambda_2 \geq \begin{cases} \Omega\left(\sqrt{d}\right) & \text{if } 1 \leq d \leq n^{2/3} \\ \Omega\left(n/d\right) & \text{if } n^{2/3} < d \leq n^{3/4} \\ \Omega\left(d^{1/3}\right) & \text{if } n^{3/4} < d \leq (1/2 - \varepsilon)n. \end{cases}$$

Theorem(B., Bucić 24): Let G be a graph n on vertices with second eigenvalue λ_2 and maximum degree $\Delta \geq 2$. Then the multiplicity of λ_2 satisfies

$$m(\lambda_2) \leq \max\left\{\frac{n}{\lambda_2^{1-o(1)}}, \frac{n}{(\log_\Delta n)^{1-o(1)}}\right\}.$$

Moreover, if $\log_\Delta n \geq \lambda_2^{O(1)}$, then $m(\lambda_2) \leq \frac{n}{\lambda_2+1}$.

Corollary(B.): Let G be a k-regular graph with second eigenvalue λ_2. If the spectral gap satisfies $k - \lambda_2 \ll n$, then

$$\lambda_2 \geq (1 - o(1))k^{1/3} .$$

Proof sketch: Starting with the adjacency matrix A, let $\alpha = \frac{1}{2\lambda_2+1}$ and define $M = (1-\alpha)I + \alpha J - 2\alpha A$.

Straightforward to check that M is positive semidefinite, so it is the Gram matrix of some unit vectors $v_1, \ldots, v_n$.

Orthogonally project $\left(\sum_{j=1}^{n} v_j\right) v_1^\intercal + v_1 \left(\sum_{j=1}^{n} v_j\right)^\intercal$ onto the span of $v_1 v_1^\intercal, \ldots, v_n v_n^\intercal$ (with respect to the Frobenius inner product).

The (Frobenius) norm of X can only decrease! $\square$

Theorem(B.): Assuming $\alpha \to 0$, we have

$$N_\alpha(r) \leq \max\left(\binom{1/\alpha^2 - 1}{2}, (2 + o(1))r\right) \cdot$$

Proof sketch: Start with the Gram matrix M of the unit vectors $v_1, \ldots, v_n$ spanning n lines. Consider its largest eigenvalue λ_1.

If $\lambda_1 \leq \frac{1}{2}\left(\frac{1}{\alpha^2} - 1\right)$, then the first bound $n \leq \binom{1/\alpha^2 - 1}{2}$ follows immediately from $n(1 + \alpha^2(n-1)) = \mathrm{tr}(M^2) \leq \lambda_1 \mathrm{tr}(M) = \lambda_1 n$.

Otherwise, we can assume that $\lambda_1 > \frac{1}{2}\left(\frac{1}{\alpha^2} - 1\right)$ and $n \geq \binom{1/\alpha^2 - 1}{2}$.

"Switching argument": negate some of the vectors so that the eigenvector x corresponding to λ_1 has all nonnegative entries.

Consider the graph with vertices $v_1, \ldots, v_n$ such that $v_i v_j$ forms an edge if and only if $\langle v_i, v_j \rangle = -\alpha$.

Theorem(B.): Assuming $\alpha \to 0$, we have

$$N_\alpha(r) \leq \max\left(\binom{1/\alpha^2-1}{2}, (2+o(1))r\right) \cdot$$

Fix i and project $\left(\sum_{j=1}^n x(j)v_j\right) v_i^\intercal + v_i \left(\sum_{j=1}^n x(j)v_j\right)^\intercal$ onto the span of $v_1v_1^\intercal, \ldots, v_nv_n^\intercal$ (with respect to the Frobenius inner product).

Since the (Frobenius) norm can only decrease, a calculation yields that the degree of v_i satisfies $d(v_i) \leq O\left(1/\alpha^3\right)$.

Using a variant of the usual Alon-Boppana theorem, we can bootstrap this bound to $d(v_i) < \frac{1}{4\alpha^2}$.

The second bound $n \leq (2+o(1))r$ then follows by applying the inequality $\mathrm{tr}(H)^2 \leq \mathrm{rk}(H)\mathrm{tr}(H^2)$ with $H = M - \alpha J$. $\square$

Theorem[Relative Bound] (Lemmens, Seidel 73): $N_\alpha(r) \le r\frac{1-\alpha^2}{1-r\alpha^2}$ for all $\alpha < \frac{1}{\sqrt{r}}$.

Proof idea: Using the Frobenius inner product, orthogonally project the $r \times r$ identity matrix I onto the span of $v_1 v_1^\intercal, \ldots, v_n v_n^\intercal$. Its length decreases from r to $\frac{n}{\alpha^2 n + 1 - \alpha^2}$.

Proof: Let $\mathscr{W} : \mathbb{R}^n \to \mathbb{R}^{r\times r}$ be the linear map given by $\mathscr{W} e_i = v_i v_i^\intercal$ (Can identify $\mathscr{W}$ with the $r^2 \times n$ matrix whose ith column is an r^2 length vectorized version of $v_i v_i^\intercal$)

Definition: Let $\mathscr{W}^\# : \mathbb{R}^{r\times r} \to \mathbb{R}^n$ denote the adjoint map with respect to the Frobenius inner product ($\langle M, \mathscr{W} v\rangle_F = \langle \mathscr{W}^\# M, v\rangle$ for all matrices $M \in \mathbb{R}^{r\times r}$ and vectors $v \in \mathbb{R}^n$).

(Can identify $\mathscr{W}^\#$ with the transpose of the matrix corresponding to $\mathscr{W}$.)

Theorem[Relative Bound] (Lemmens, Seidel 73): $N_\alpha(r) \leq r\frac{1-\alpha^2}{1-r\alpha^2}$ for all $\alpha < \frac{1}{\sqrt{r}}$.

Proof(continued): Observe that $\mathscr{P} = \mathscr{W}(\mathscr{W}^\#\mathscr{W})^{-1}\mathscr{W}^\#$ denotes the orthogonal projection onto the span of $v_1v_1^\intercal, \ldots, v_nv_n^\intercal$.

(Easy to verify that $\mathscr{P}^2 = \mathscr{P} = \mathscr{P}^\#$ and $\mathscr{P}\mathscr{W} = \mathscr{W}$)

Now compute $\langle I, I\rangle_F = \mathrm{tr}(I^\intercal I) = r$ and

$$\langle \mathscr{P}I, \mathscr{P}I\rangle_F = I^\#\mathscr{P}^\#\mathscr{P}I = I^\#\mathscr{P}I = I^\#\mathscr{W}(\mathscr{W}^\#\mathscr{W})^{-1}\mathscr{W}^\#I.$$

By definition, $(\mathscr{W}^\#I)_i = \langle v_iv_i^\intercal, I\rangle_F = \mathrm{tr}(v_iv_i^\intercal I) = 1$ so $\mathscr{W}^\#I = \mathbb{1}$.

Moreover $(\mathscr{W}^\#\mathscr{W})_{i,j} = \left\langle v_iv_i^\intercal, v_jv_j^\intercal\right\rangle_F = \langle v_i, v_j\rangle^2$ so that $\mathscr{W}^\#\mathscr{W} = (1-\alpha^2)I + \alpha^2 J$ and $(\mathscr{W}^\#\mathscr{W})^{-1} = \frac{1}{1-\alpha^2}\left(I - \frac{1}{n+1/\alpha^2+1}J\right)$

Thus $\langle \mathscr{P}I, \mathscr{P}I\rangle_F = \frac{1}{1-\alpha^2}\mathbb{1}^\intercal\left(I - \frac{1}{n+1/\alpha^2-1}J\right)\mathbb{1} = \frac{n}{\alpha^2 n+1-\alpha^2}.$ $\square$

New results in the complex setting

Given a pair of complex lines $U, V \subset \mathbb{C}^r$, the quantity $|\langle u, v\rangle|$ is the same for all unit vectors $u \in U, v \in V$ and so $\arccos|\langle u, v\rangle|$ is called the **Hermitian angle** between U and V.

We define $N_\alpha^{\mathbb{C}}(r)$ to be the maximum number of complex equiangular lines in $\mathbb{C}^r$ with common Hermitian angle $\arccos(\alpha)$.

Theorem[Absolute bound] (Delsarte, Goethals, Seidel 75): $N_\alpha^{\mathbb{C}}(r) \leq r^2$.

Conjecture (Zauner 99): For each $r \in \mathbb{N}$, $\max_\alpha N_\alpha^{\mathbb{C}}(r) = r^2$ and a construction can be obtained as the orbit of a vector under the action of a Weyl-Heisenberg group.

Collections of r^2 complex equiangular lines in $\mathbb{C}^r$ are known as SIC-POVMs/SICs in quantum information theory.

New results in the complex setting

Theorem[Relative Bound] (Delsarte, Goethals, Seidel 75)**:**
$N_\alpha^{\mathbb{C}}(r) \leq r\,\frac{1-\alpha^2}{1-r\alpha^2}$ for all $r \leq 1/\alpha^2 - 1$.

Theorem(B.)**:** If $r \leq \frac{1-o(1)}{\alpha^3}$, then $N_\alpha^{\mathbb{C}}(r) \leq \left(\frac{1}{\alpha^2} - 1\right)^2$, with equality if and only if there exists a SIC in $1/\alpha^2 - 1$ dimensions.

Otherwise $N_\alpha^{\mathbb{C}}(r) \leq \frac{1+\alpha}{\alpha} r + O\left(\frac{1}{\alpha^3}\right)$.

Future directions for research

- Unit vectors corresponding to equiangular lines are equivalently spherical $\{\alpha, -\alpha\}$-codes. Extend methods to more general spherical L-codes.
- Determine $N_\alpha^{\mathbb{C}}(r)$ up to a multiplicative constant.
- Generalize to other graph matrices (ex: Laplacian).
- Generalize to equiangular subspaces.
- Generalize to signed graphs and unitarily-signed graphs.


thank you
danke
謝謝
ngiyabonga
teşekkür ederim
gracias
merci
obrigado
bedankt
dziękuję
go raibh maith agat
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sukriya
kop khun krap
terima kasih
grazie
arigatô
dakujem
takk
tapadh leat
dank je
спасибо
Баярлалаа
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