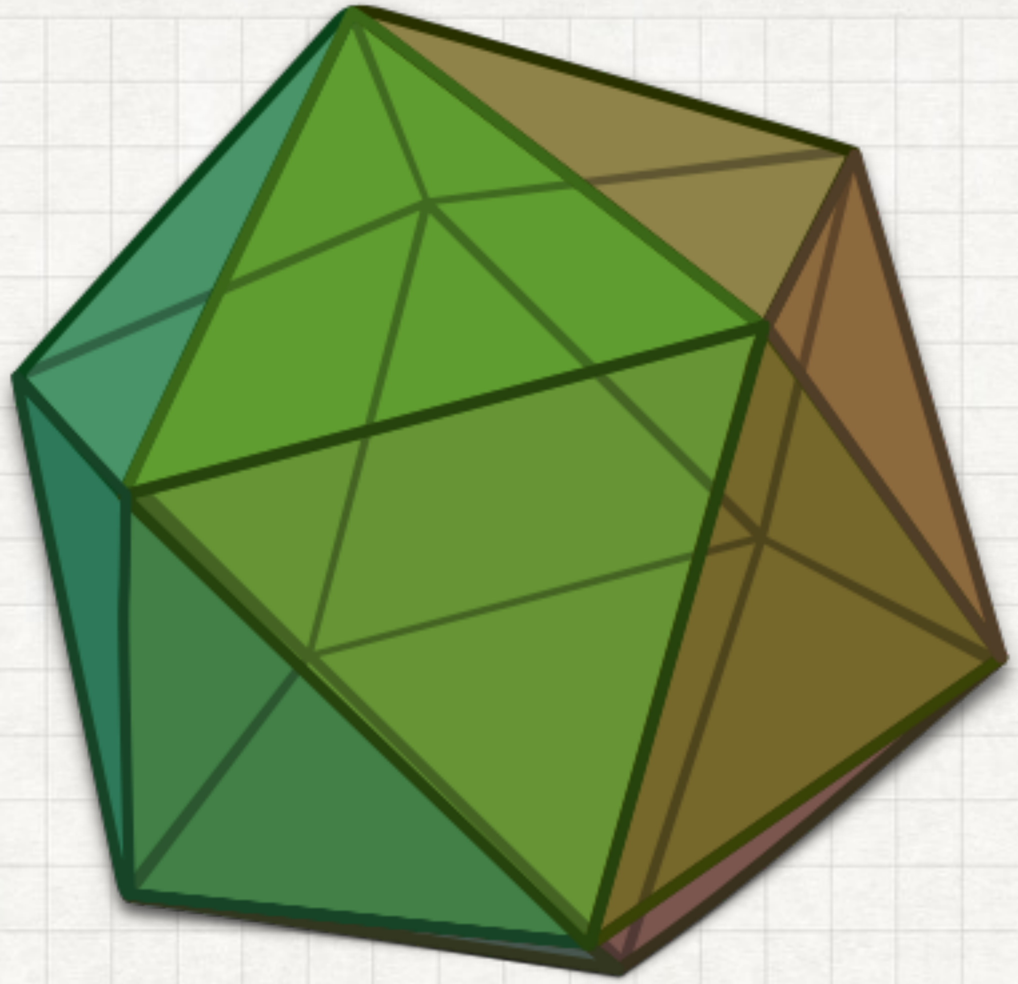

EQUIANGULAR LINES AND SUBSPACES IN EUCLIDEAN SPACES

By: Igor Balla

Joint work with: Felix Dräxler, Peter Keevash, Benny Sudakov

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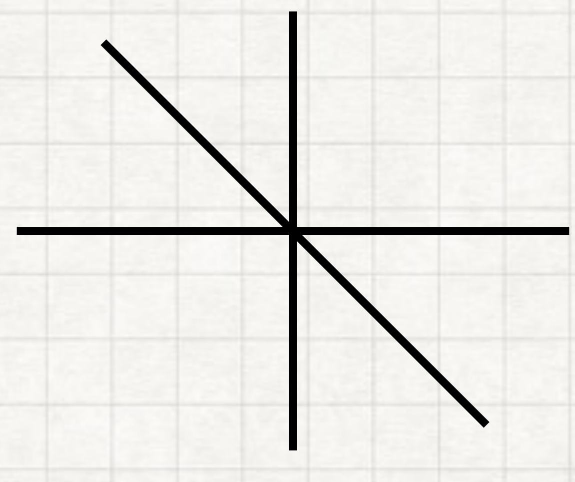

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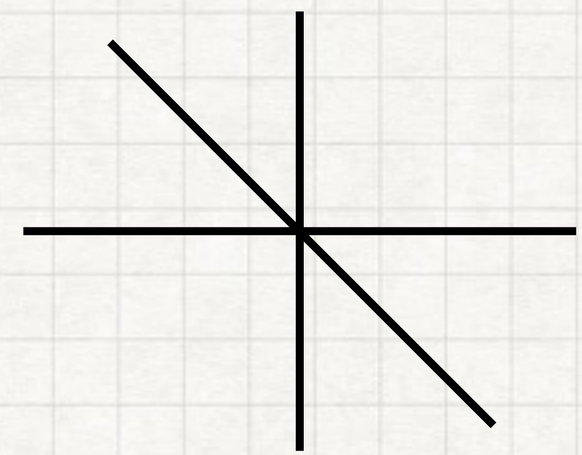

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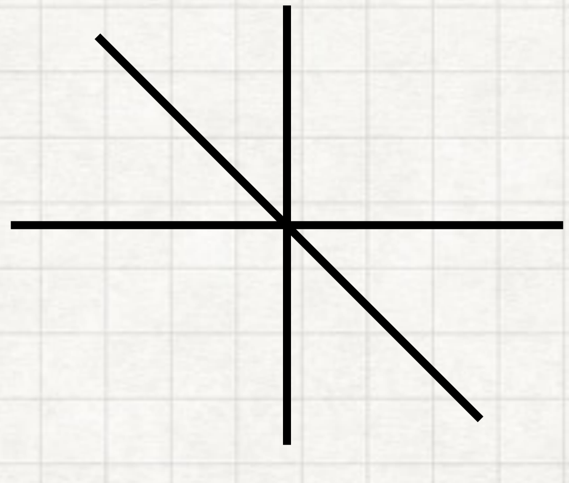

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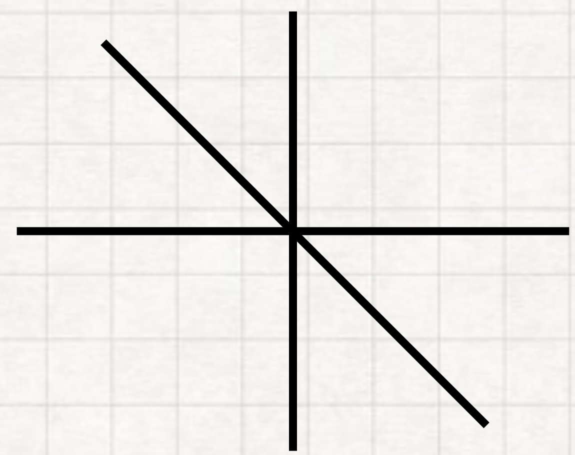

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Earliest work:

Haantjes, Seidel 47-48

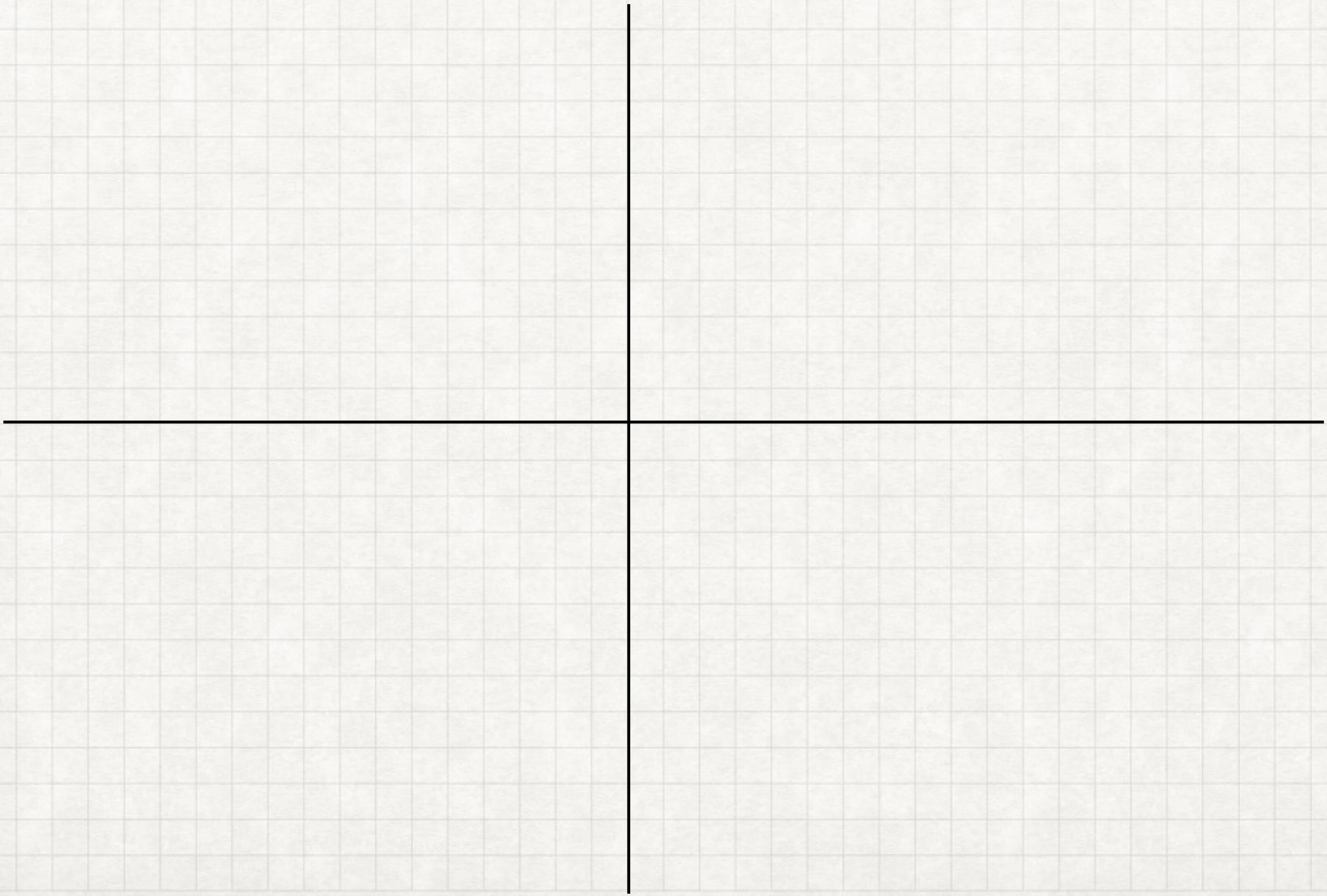
Blumenthal 49

Van Lint, Seidel 66

Lemmens, Seidel 73

…

Examples



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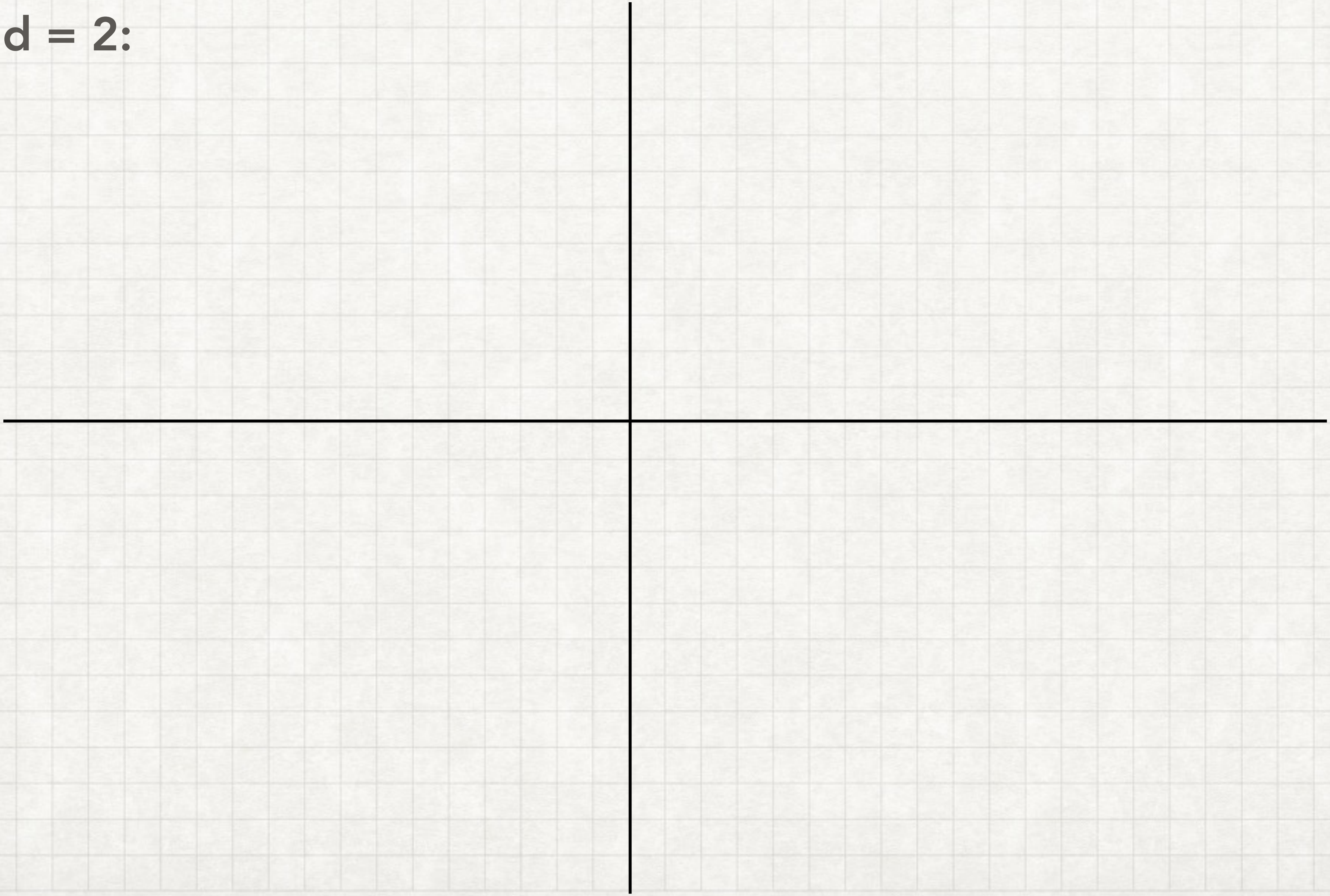

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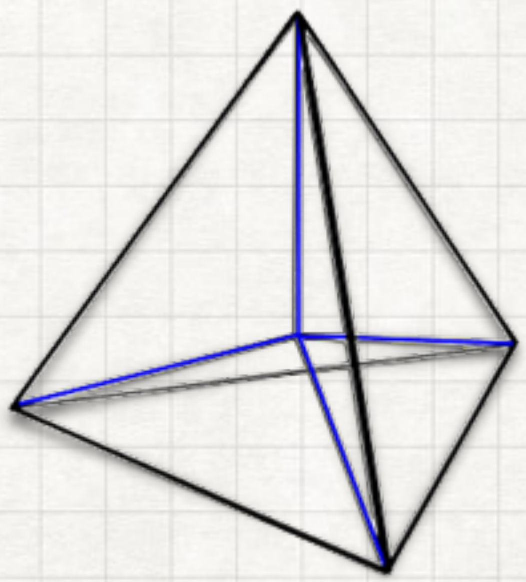

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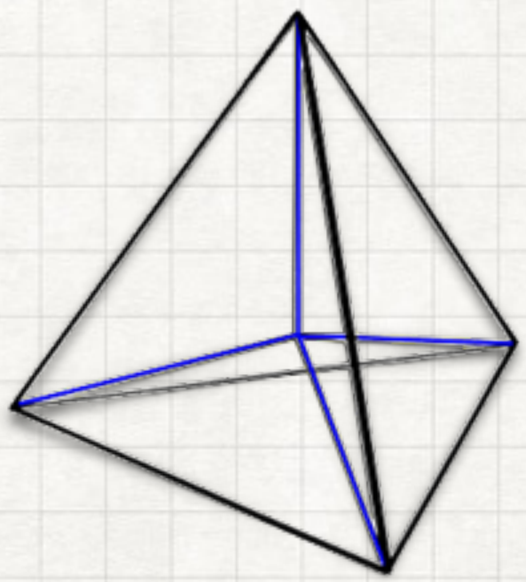

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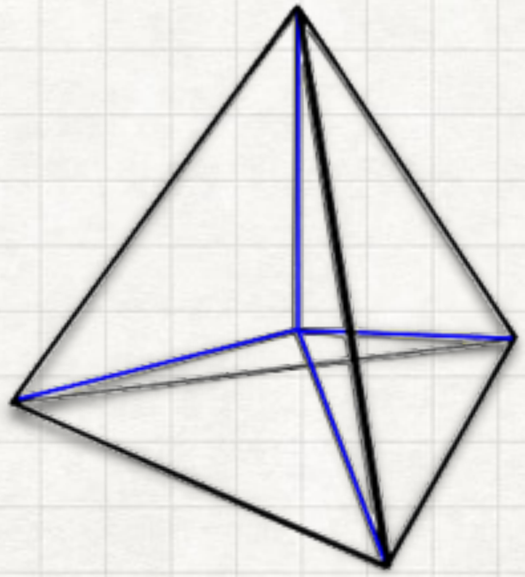
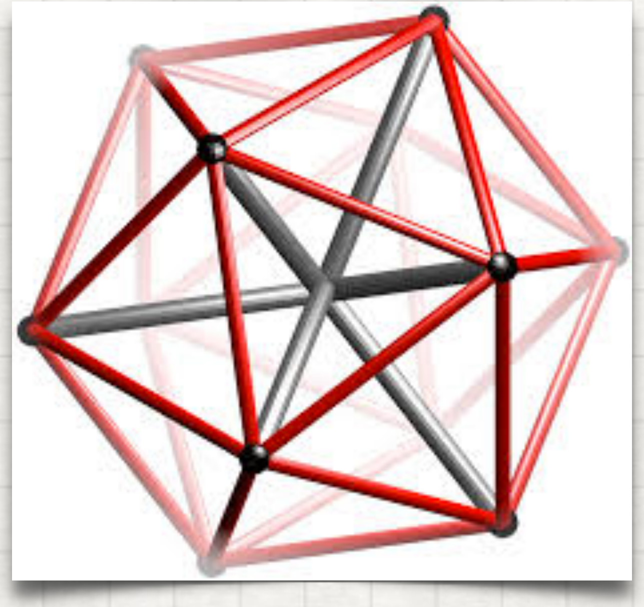


d = 3: Icosahedron

6 lines



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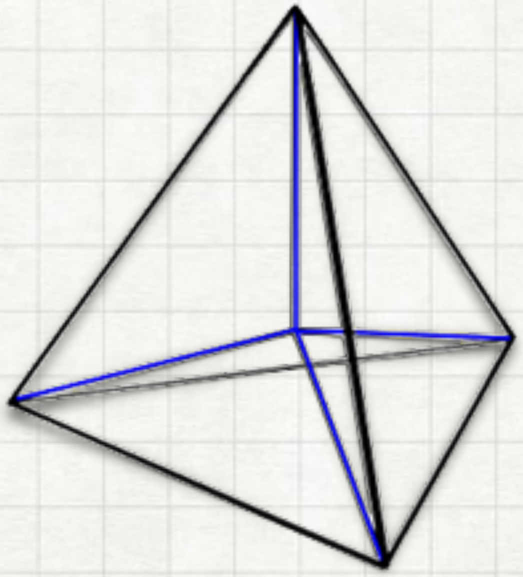

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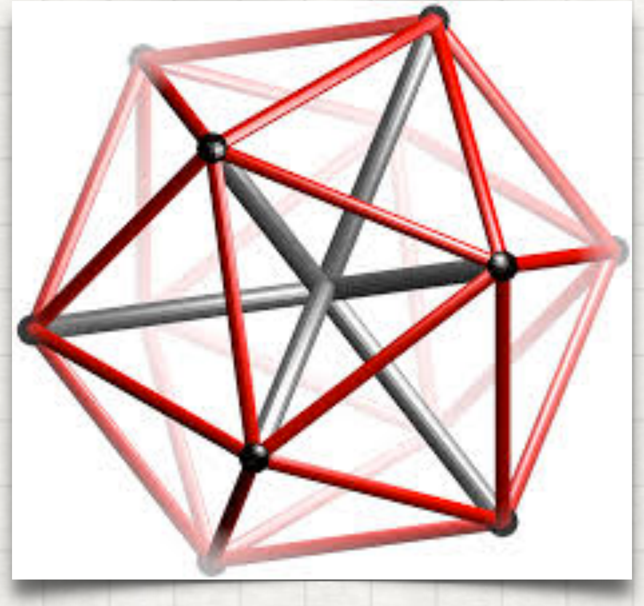

d = 7:

Examples

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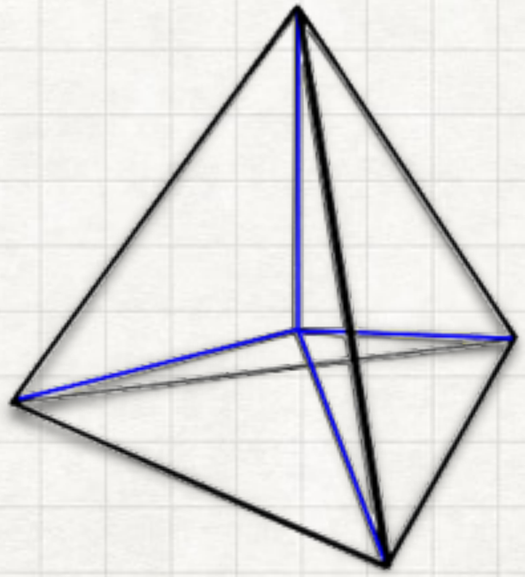

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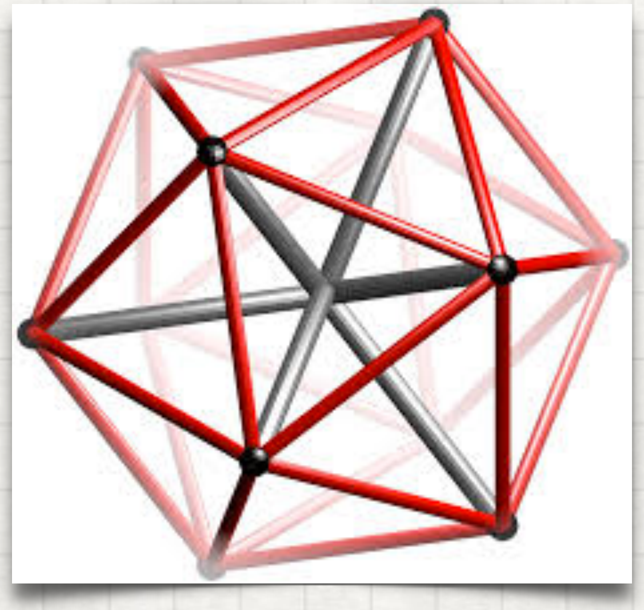

d = 7:

28 lines

Take all 28 permutations of the vector (3,3,-1,-1,-1,-1,-1,-1).

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$$\begin{cases} = 2d-2 & \text{if } \alpha \text{ is } 1/3 \\ \leq 1.93d & \text{otherwise.} \end{cases}$$

Ideas behind the upper bound

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Definition: Call the edge $\{x_i, x_j\}$ red if $x_i \cdot x_j = +\alpha$ and call it blue if $x_i \cdot x_j = -\alpha$. So we get a red-blue edge colored complete graph G on n vertices.

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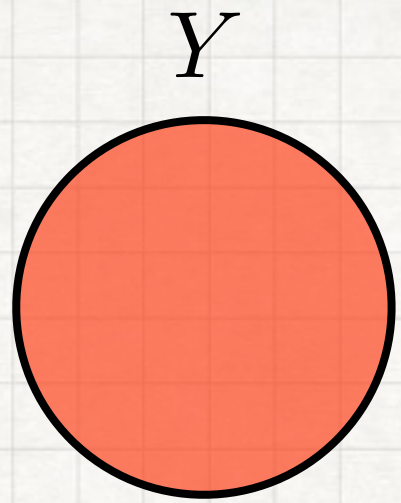
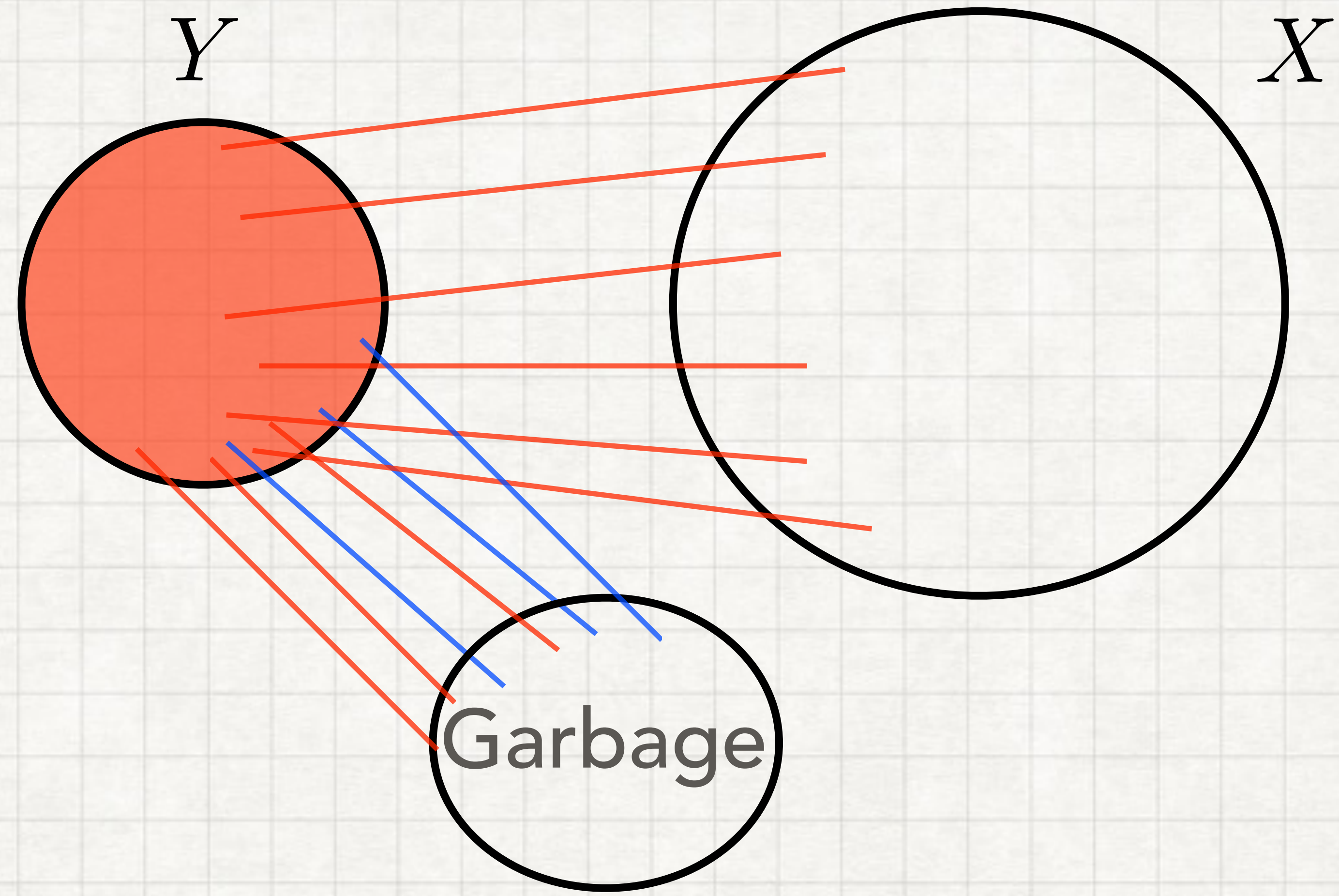
Thus by Ramsey's theorem it has a large red clique Y!


Y

Most of the remaining vertices X connect to Y entirely via red edges.



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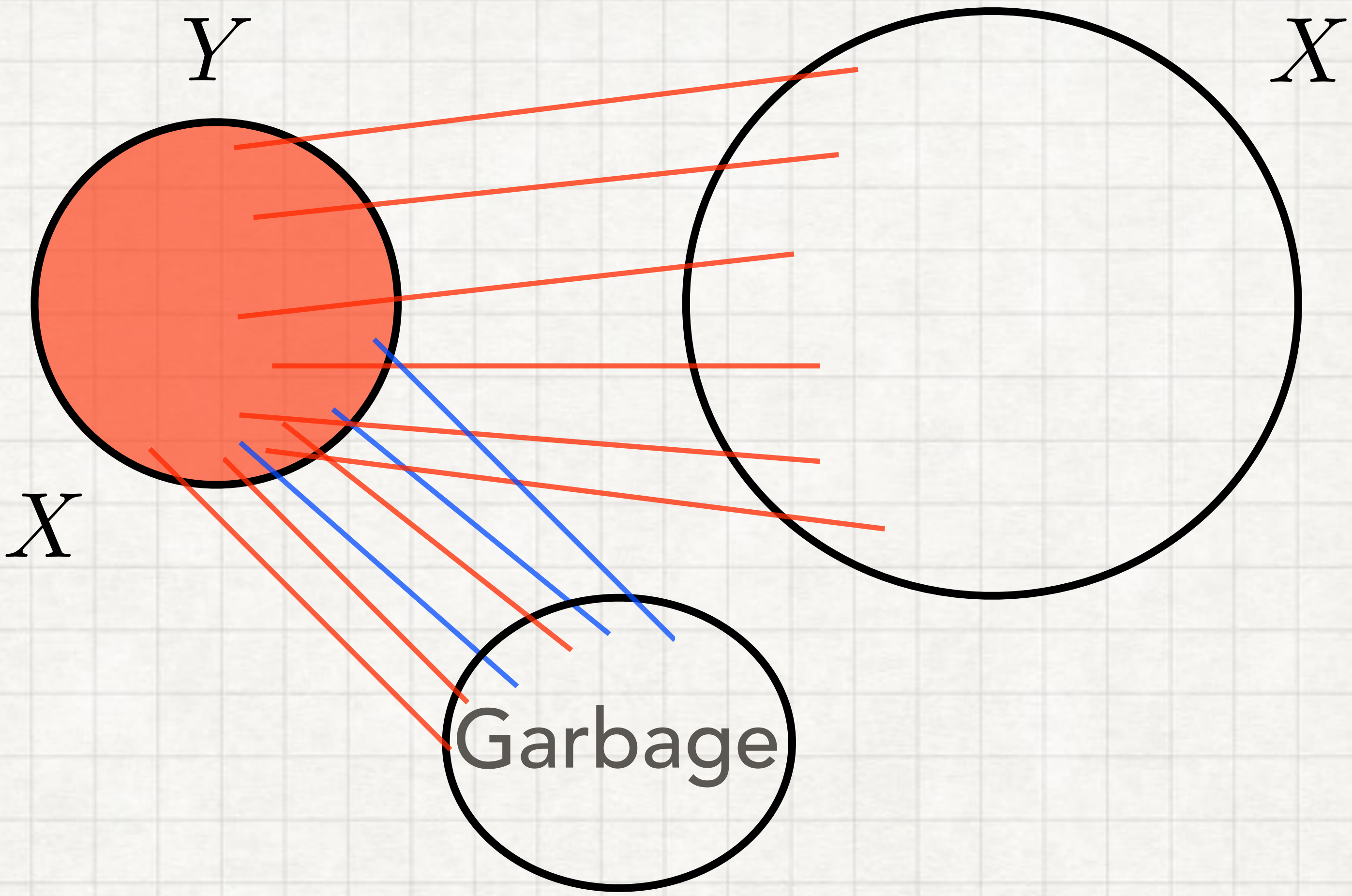
Now project the vectors of X onto the orthogonal complement of the span of the vectors of Y.



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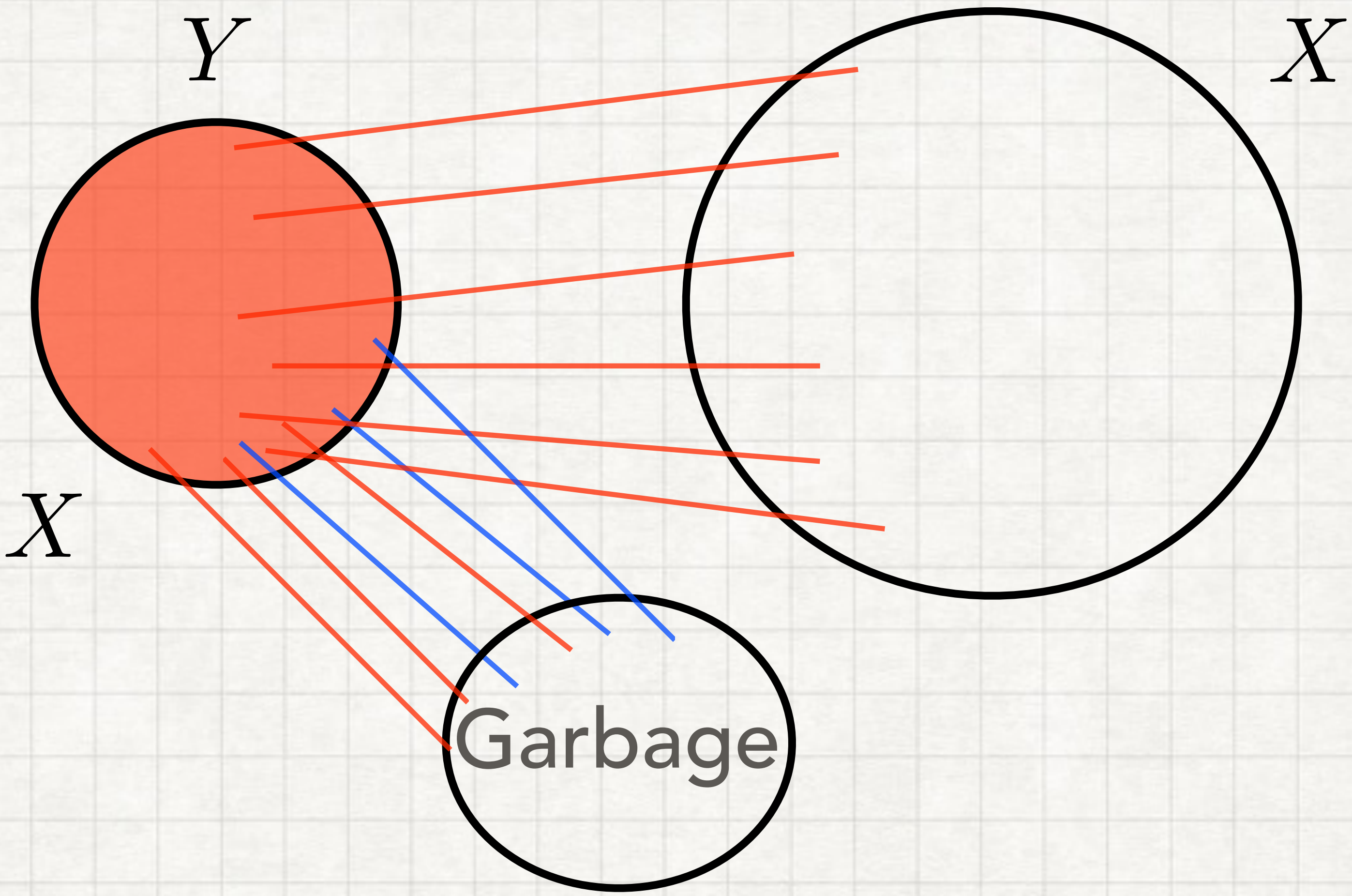

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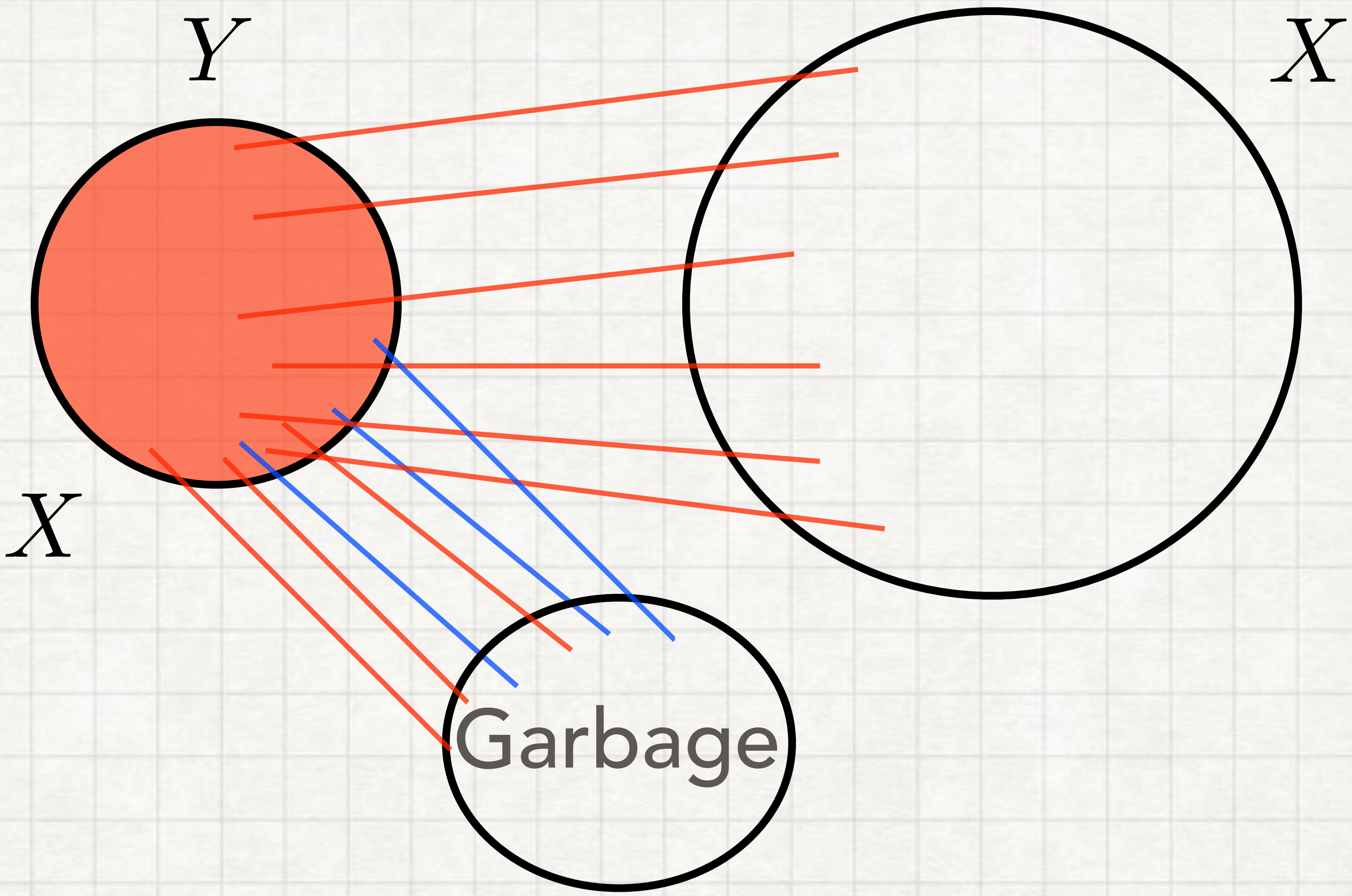

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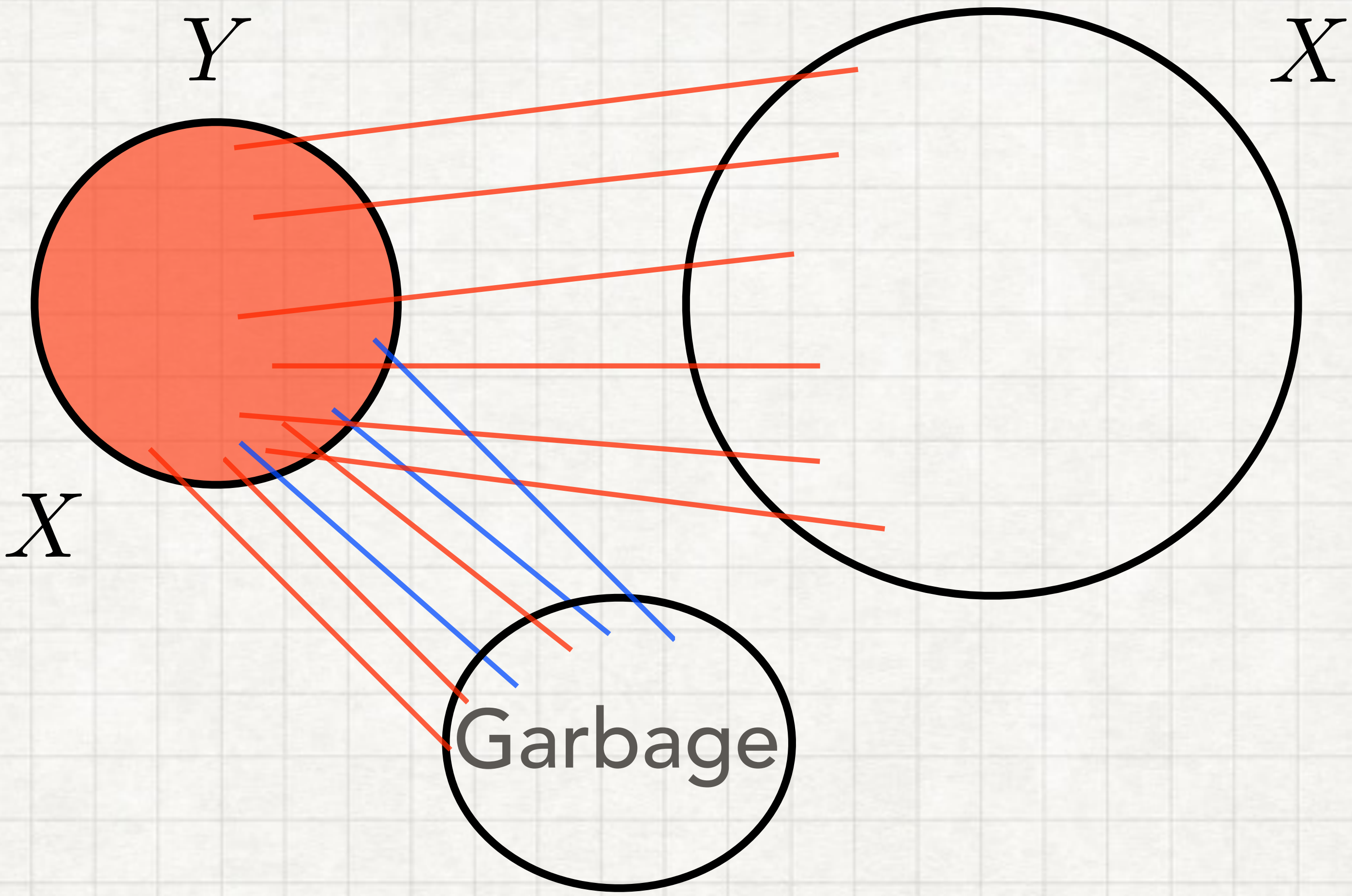

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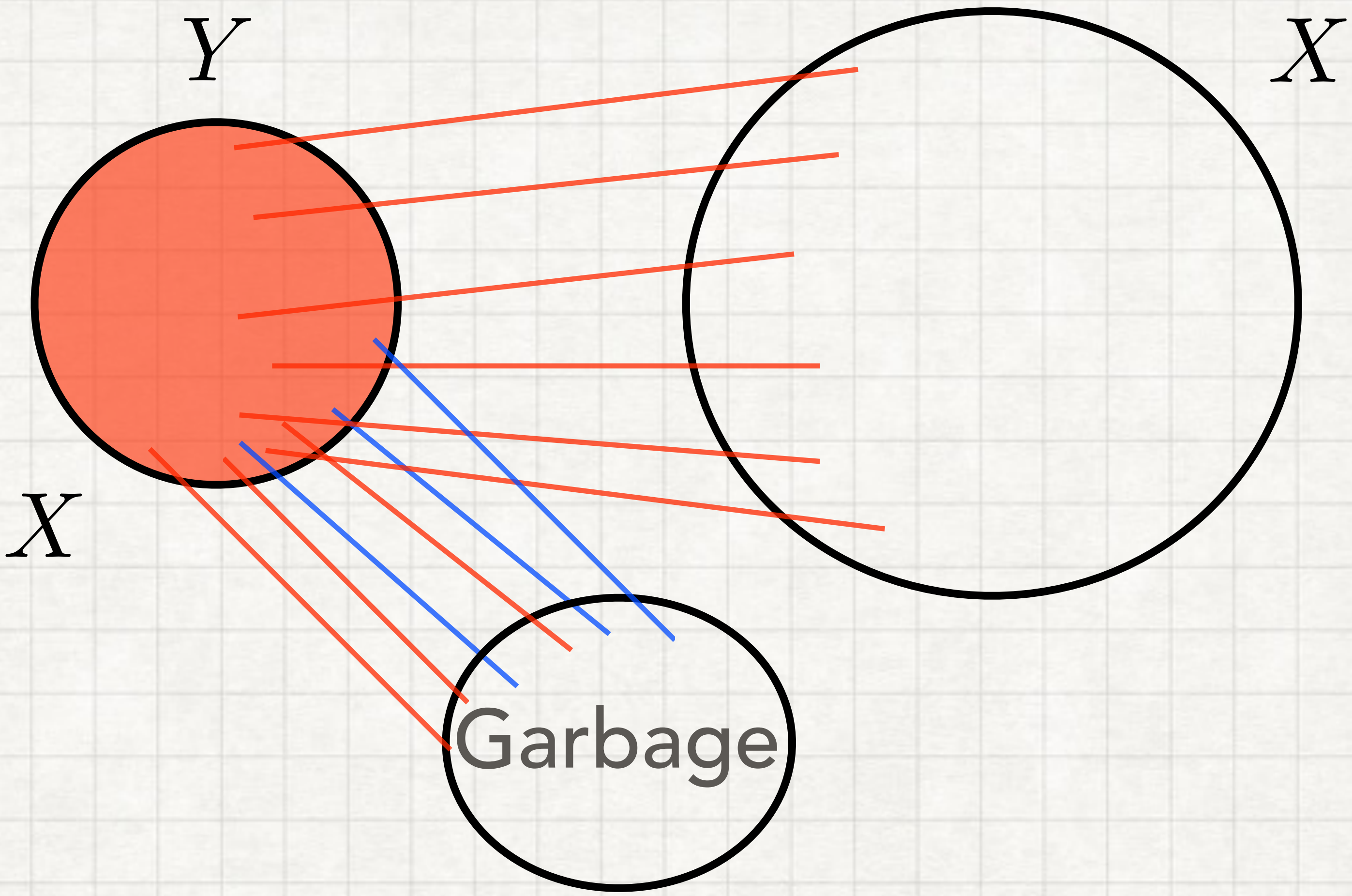

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Lemma (Schnirelmann 30 / Bellman 60 / Alon '09…):

For any symmetric matrix M with rank d, $\operatorname{tr}(M)^2 \leq d \operatorname{tr}(M^2)$.

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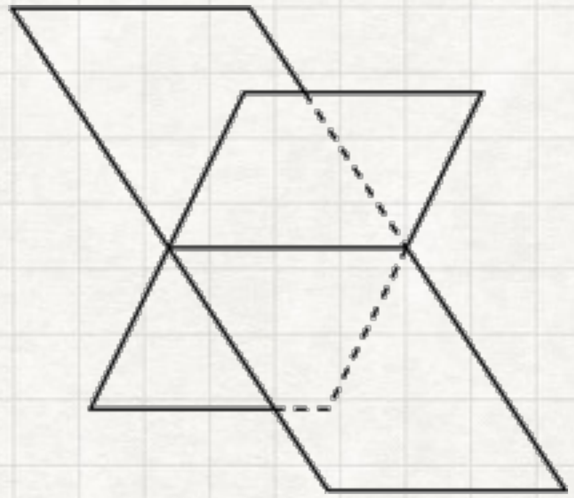

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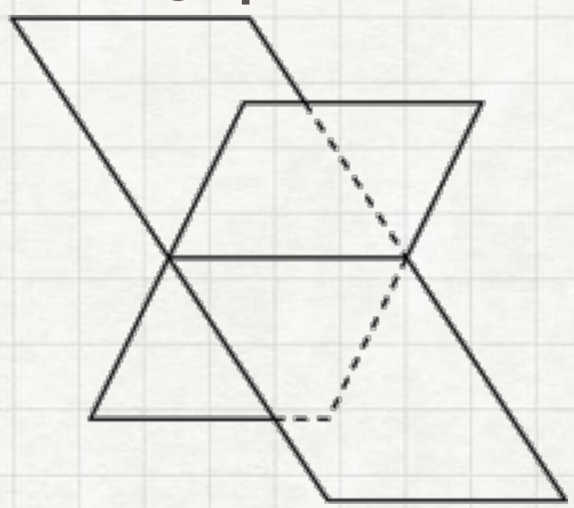

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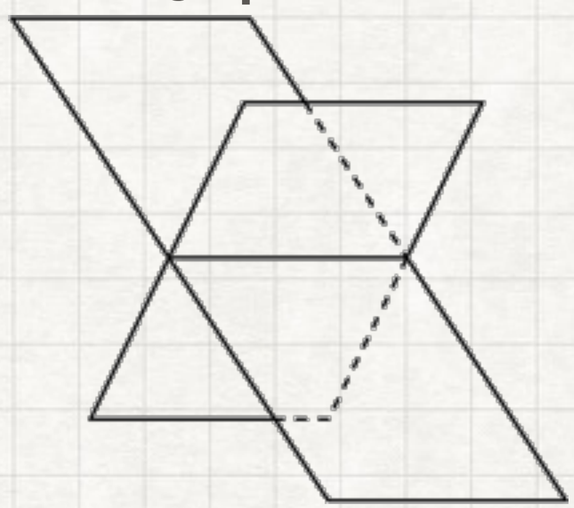

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We represent a k-dimensional subspace of $\mathbb{R}^n$ by an $n \times k$ matrix U whose columns form an orthonormal basis for the subspace.

Full answer: There are k principal angles $0 \leq \theta_1 \leq \ldots \theta_k \leq \pi/2$ between U and V, defined by $\theta_i = \arccos \sqrt{\lambda_i}$ where $1 \geq \lambda_1 \geq \ldots \geq \lambda_k \geq 0$ are the eigenvalues of $V^T U U^T V$.

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and hence $f_1, \dots, f_n$ are linearly independent. $\square$

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DONE