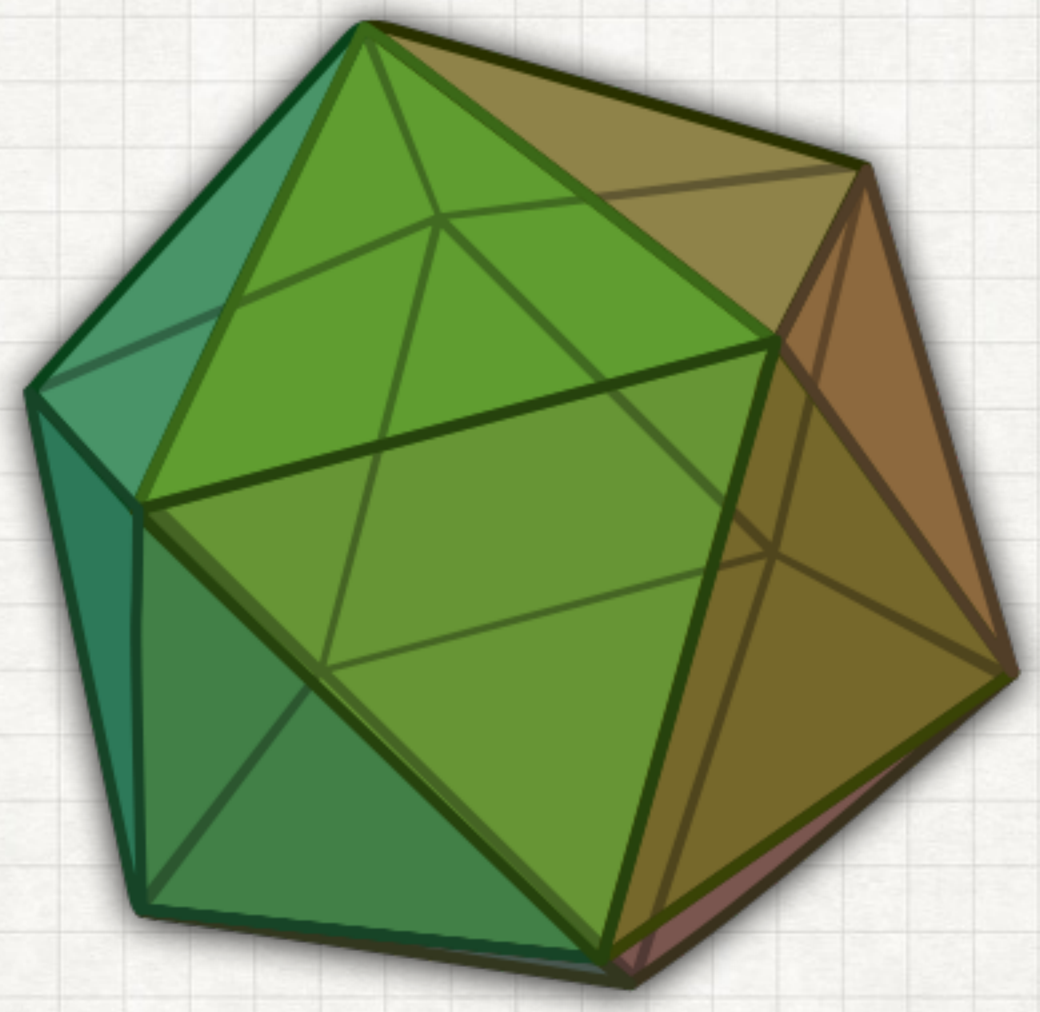

EQUIANGULAR LINES

By: Igor Balla

Joint work with: Felix Dräxler, Peter Keevash, Benny Sudakov

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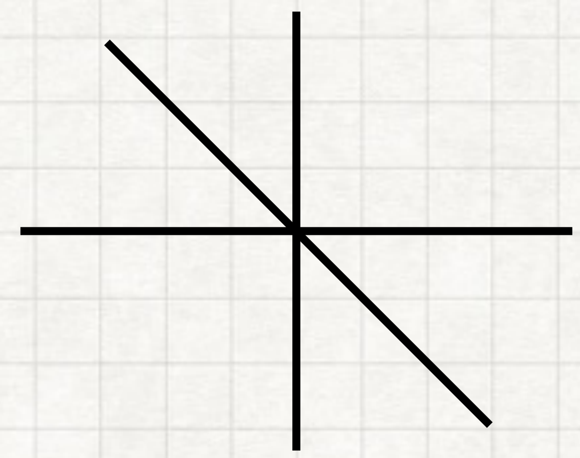

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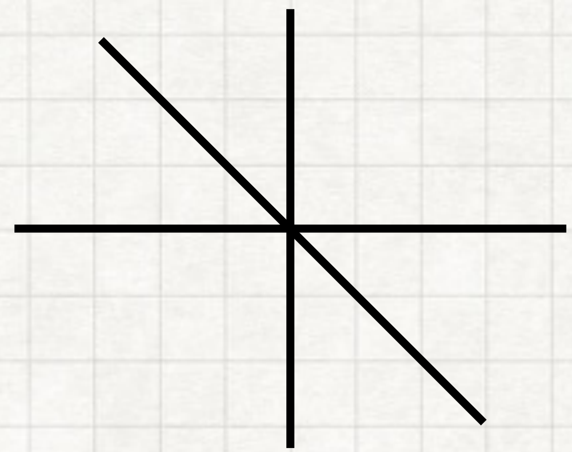

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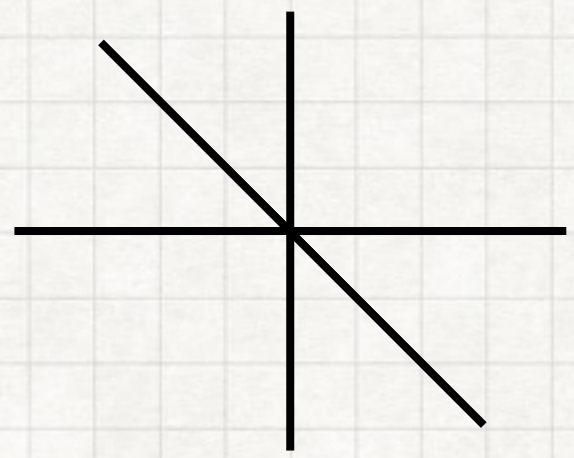

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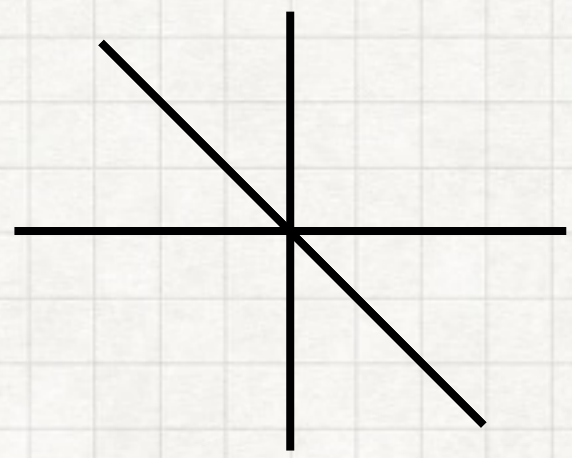

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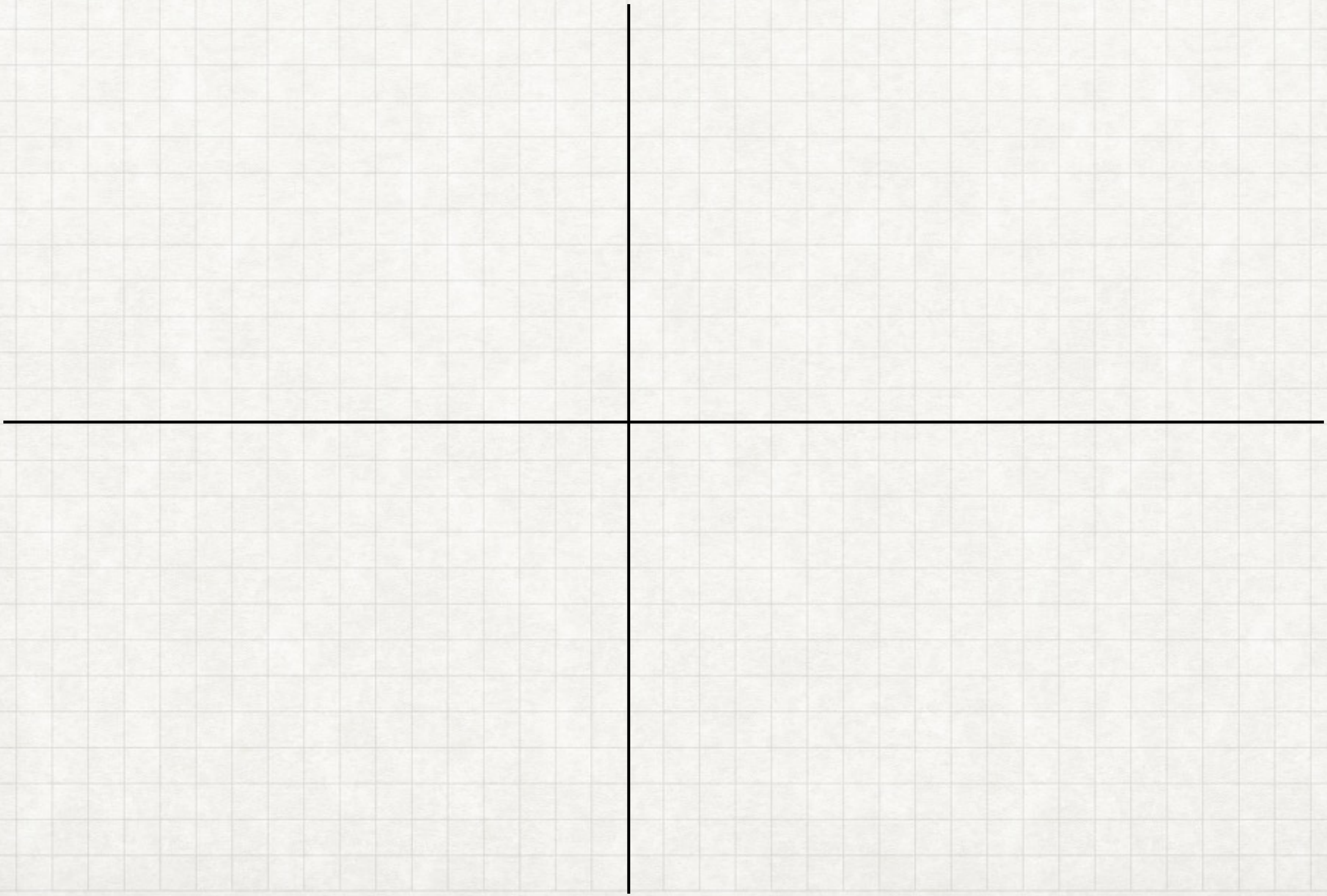
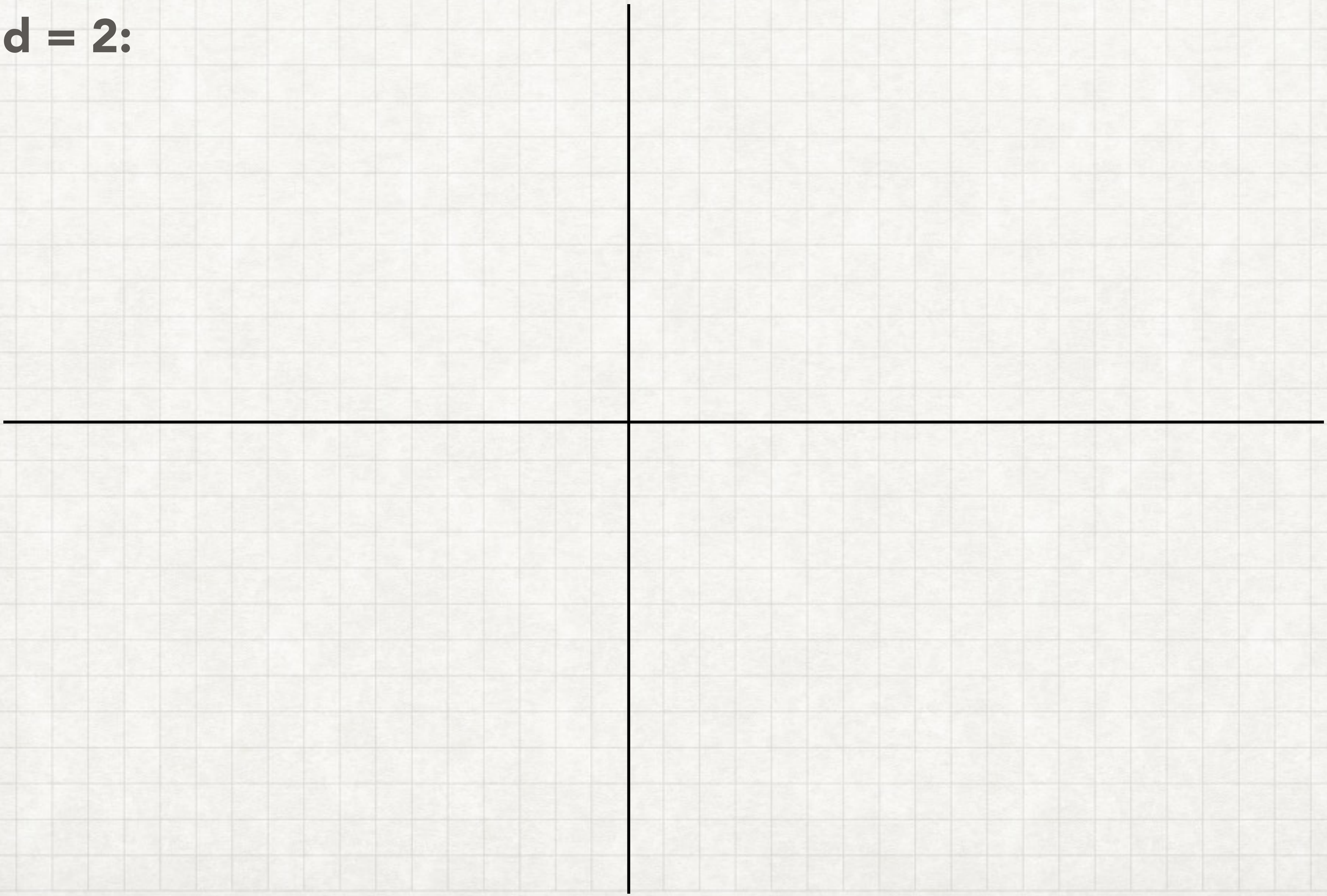
For d = 2, 3 Greeks?

Examples



Examples

d = 2:



Examples

d = 2: Triangle

3 lines

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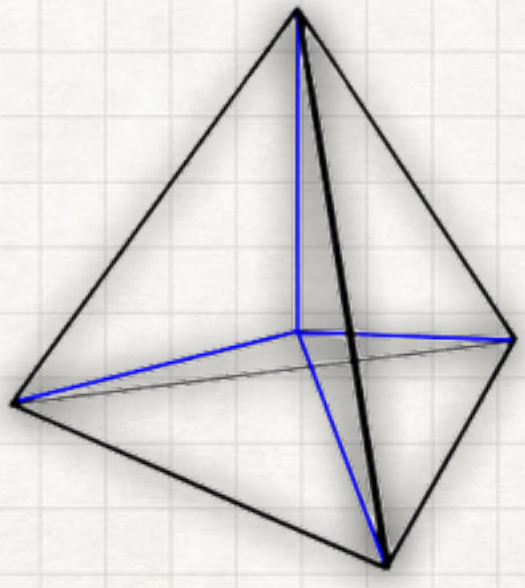
In general, d-simplex gives d+1 lines:



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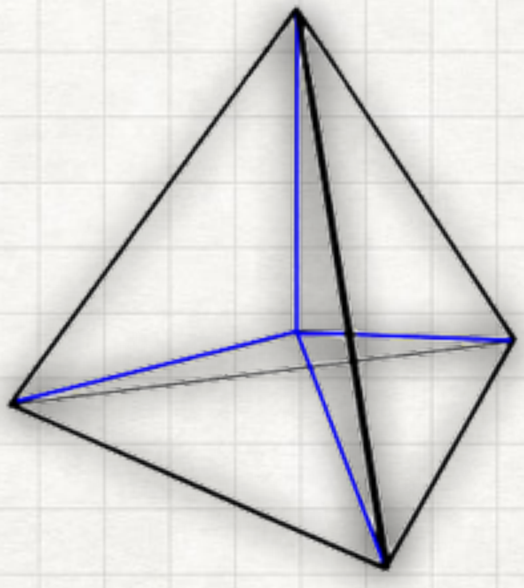

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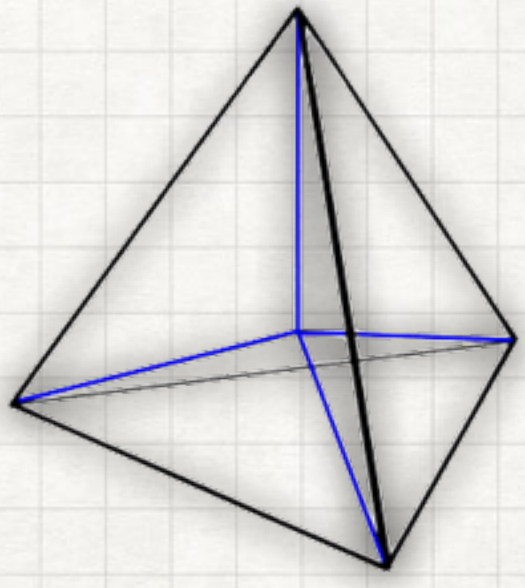

d = 3: Icosahedron

6 lines

Examples

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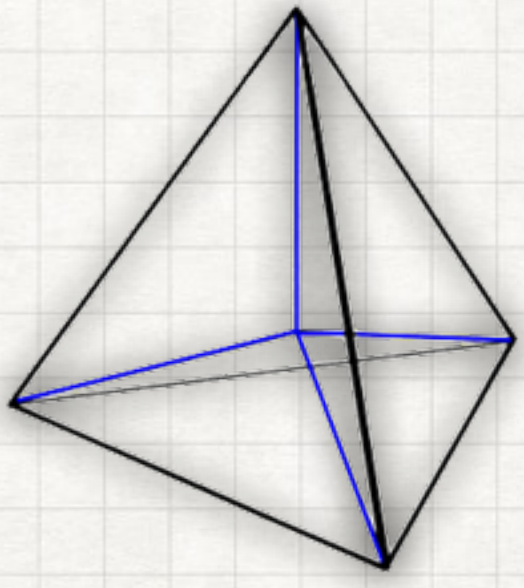

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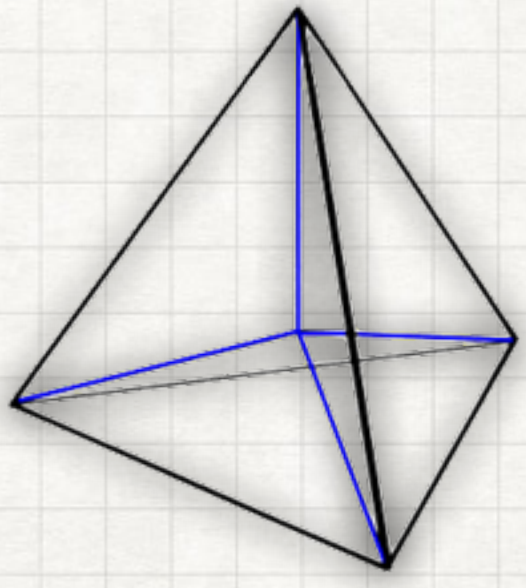

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28 lines

Take all 28 permutations of the vector (3,3,-1,-1,-1,-1,-1,-1).

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Question: Can we have $\Omega(d^2)$ lines in $\mathbb{R}^d$?

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Question: What if the angle is fixed, i.e. doesn't go to zero with d?

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eigenvalue	multiplicity
$\frac{2}{3}(d-1)$	1
$4/3$	$d-1$
0	$d-2$

$$\overbrace{}^{2d-2} \qquad 2d-2 \left(\begin{array}{ccc} \boxed{\begin{matrix} 1 & -1/3 \\ -1/3 & 1 \end{matrix}} & & 1/3 \\ & \ddots & \\ 1/3 & & \boxed{\begin{matrix} 1 & -1/3 \\ -1/3 & 1 \end{matrix}} \end{array} \right)$$

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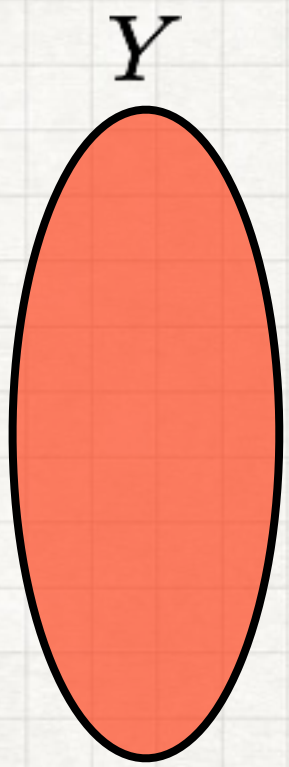
Thus by Ramsey's theorem it has a large red clique Y. Note that we can take $|Y| \to \infty$ as slowly as we need.

Strategy: Try to bound the remaining vertices based on how they connect to Y.



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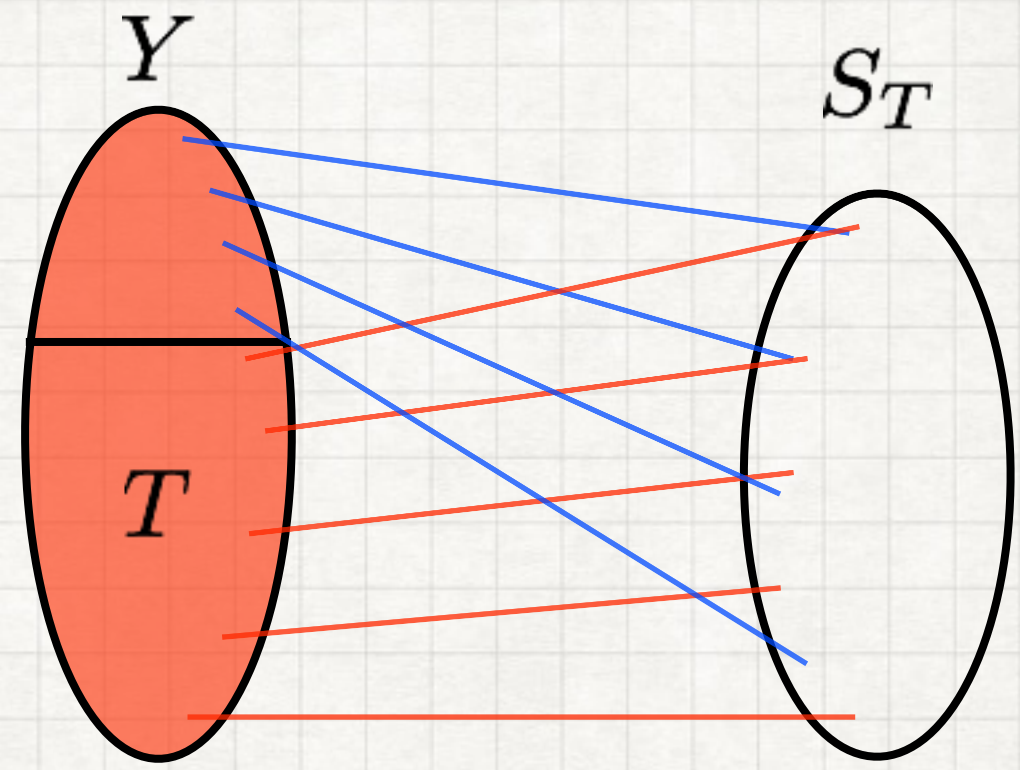

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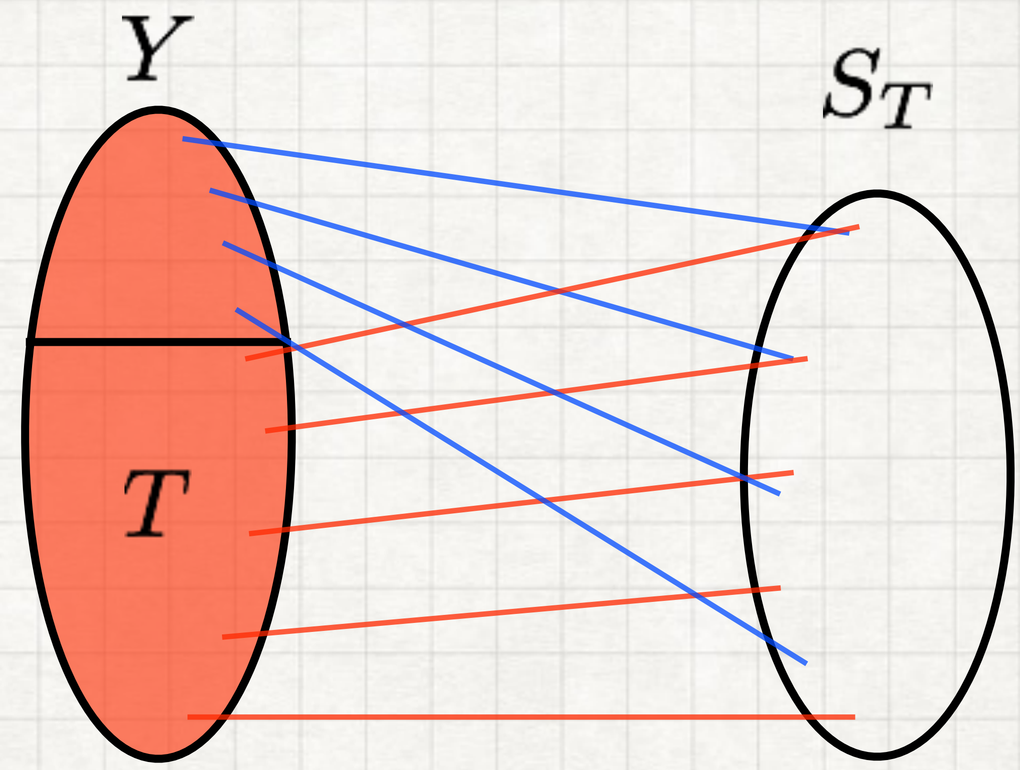
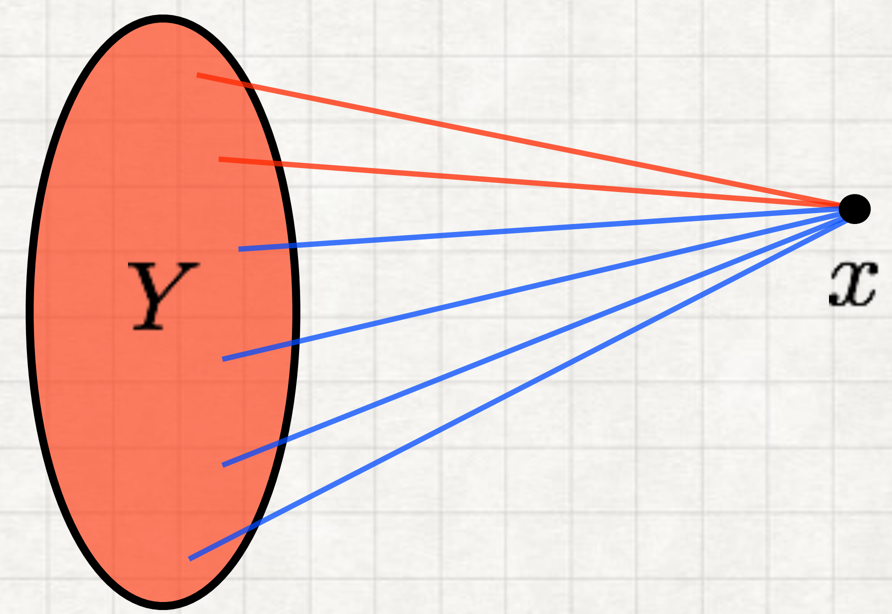


If $x \in G \backslash Y$ has more blue than red edges to Y, then negate x.



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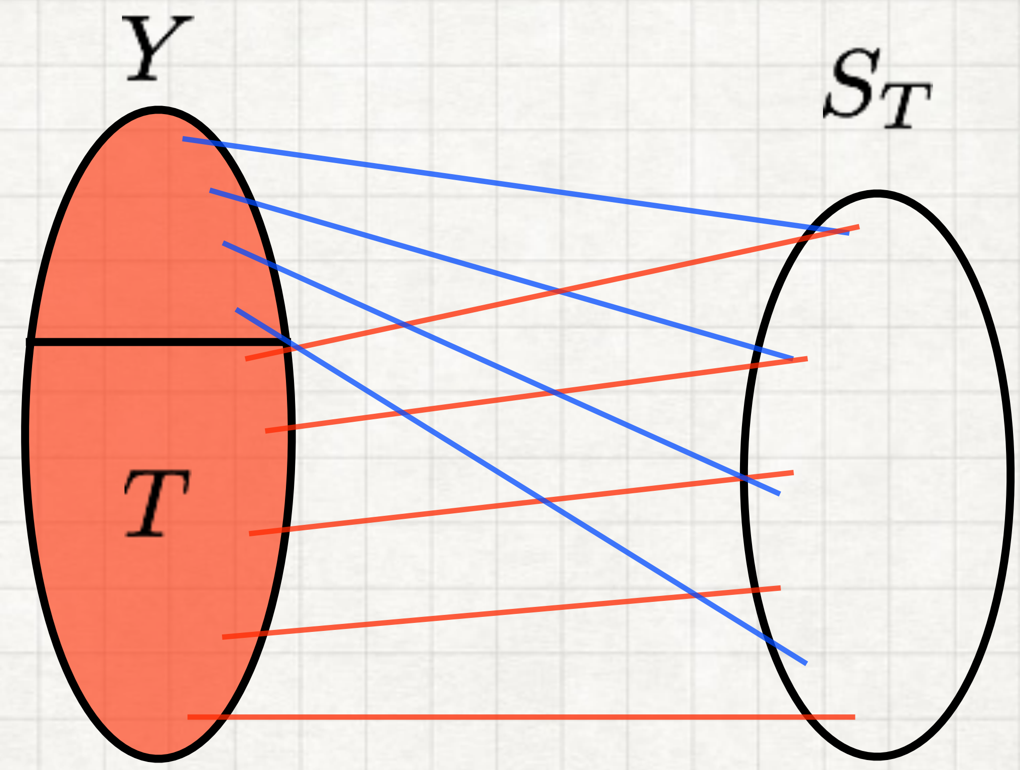

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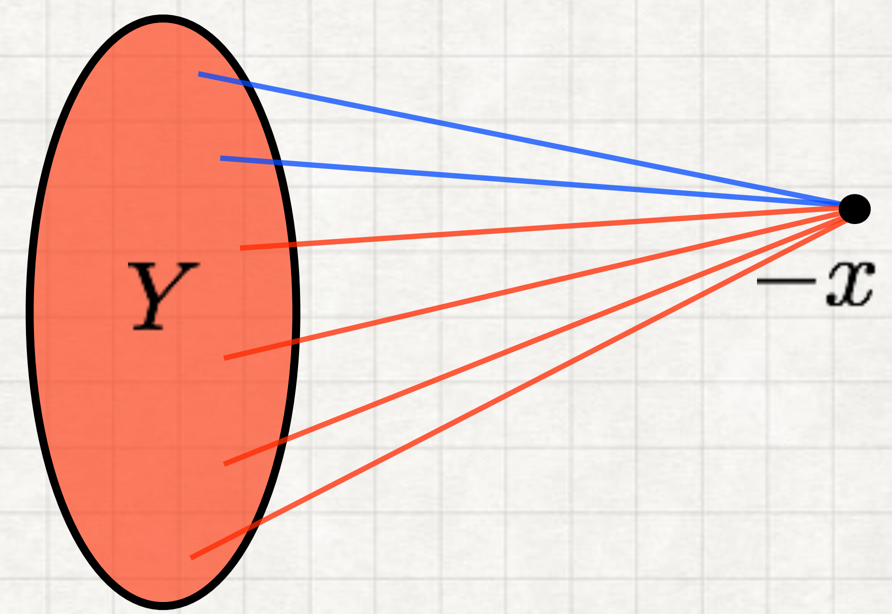
The edge colors flip so $-x$ has more red than blue edges to Y.



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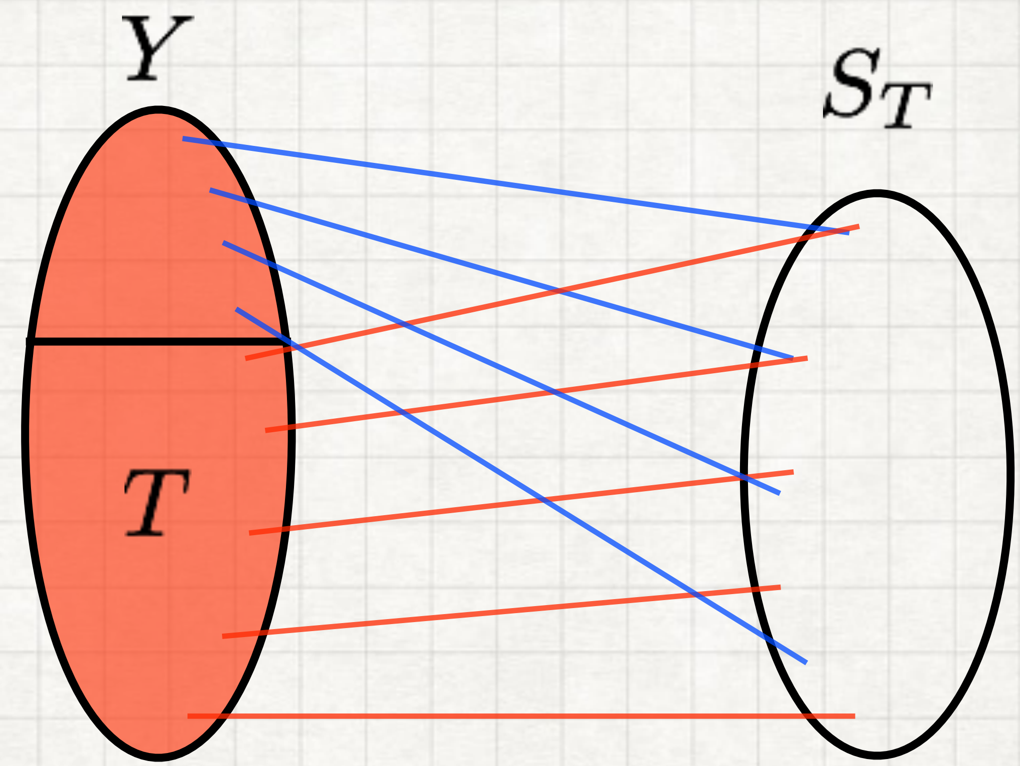

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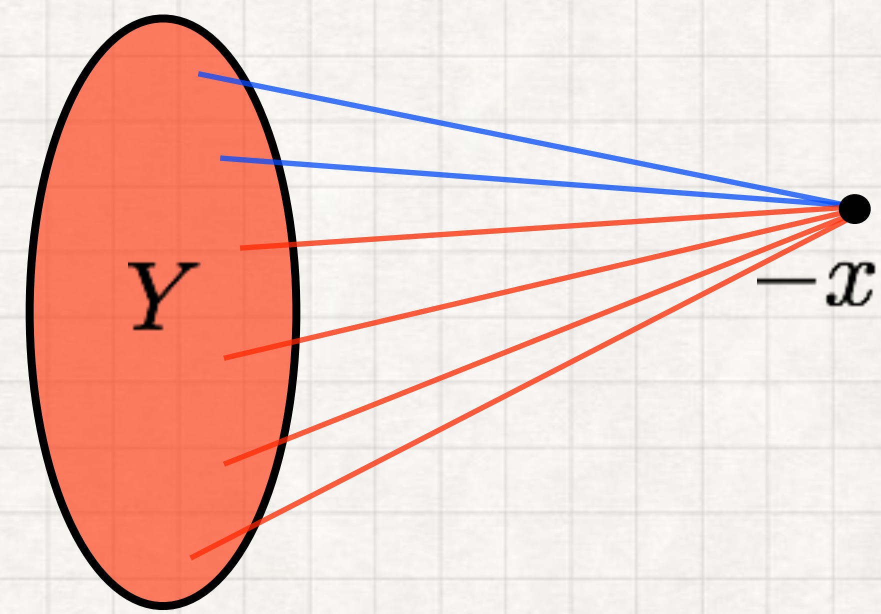
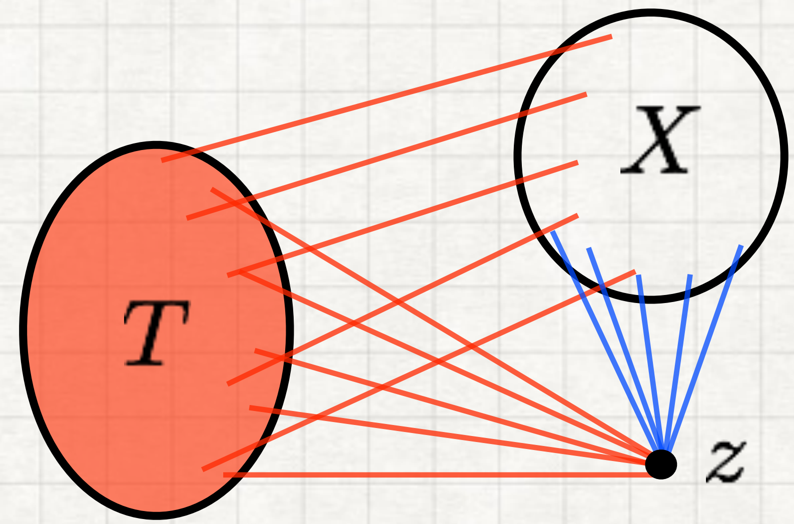


This makes $S_T = \emptyset$ for all $|T| < |Y|/2$. Otherwise we have $|T| \geq |Y|/2 \to \infty$.

Orthogonal Projection

Lemma 2: If T is a red clique with $|T| \to \infty$ and X, z are such that all edges from T to $X \cup \{z\}$ are red and all edges from z to X are blue, then

$$|X| \leq \frac{1}{\beta^2} + o(1) \quad \text{where} \quad \beta := \frac{2\alpha}{1-\alpha}.$$



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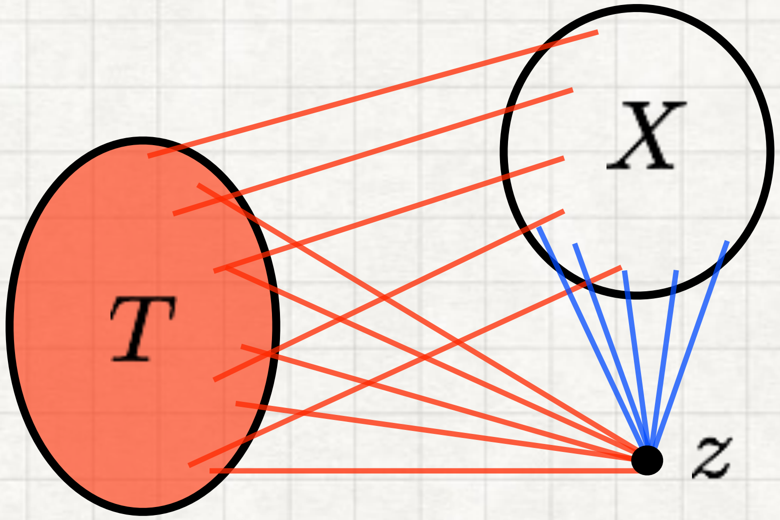

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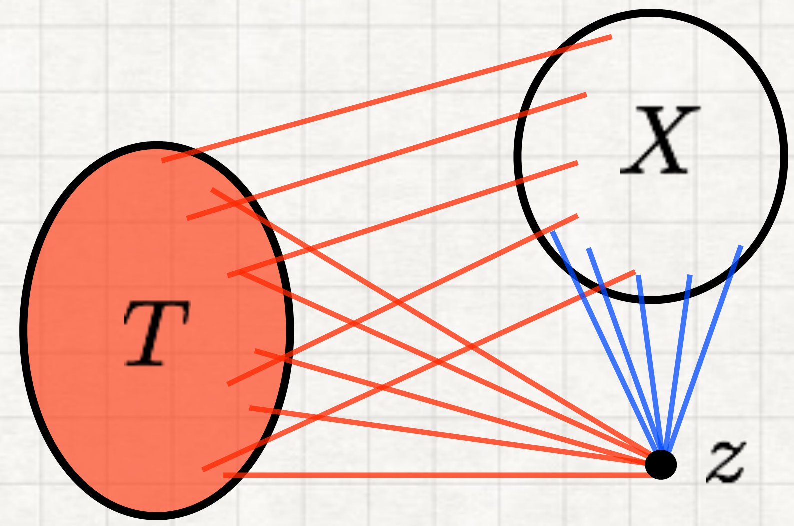

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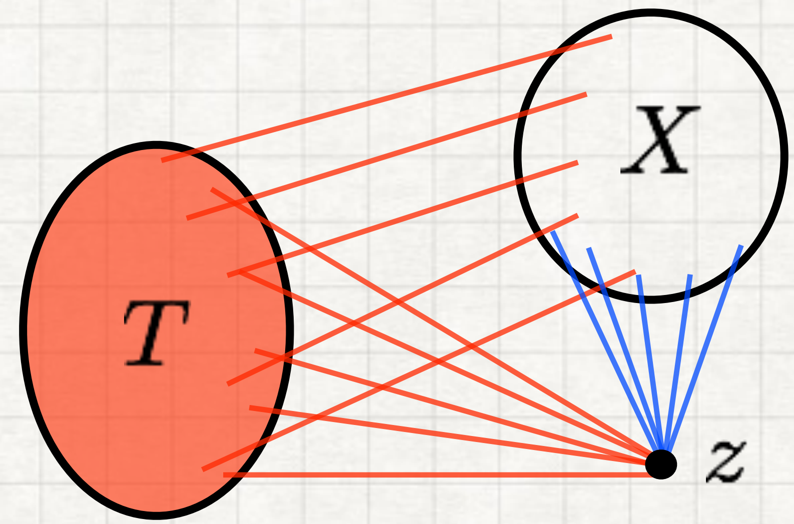

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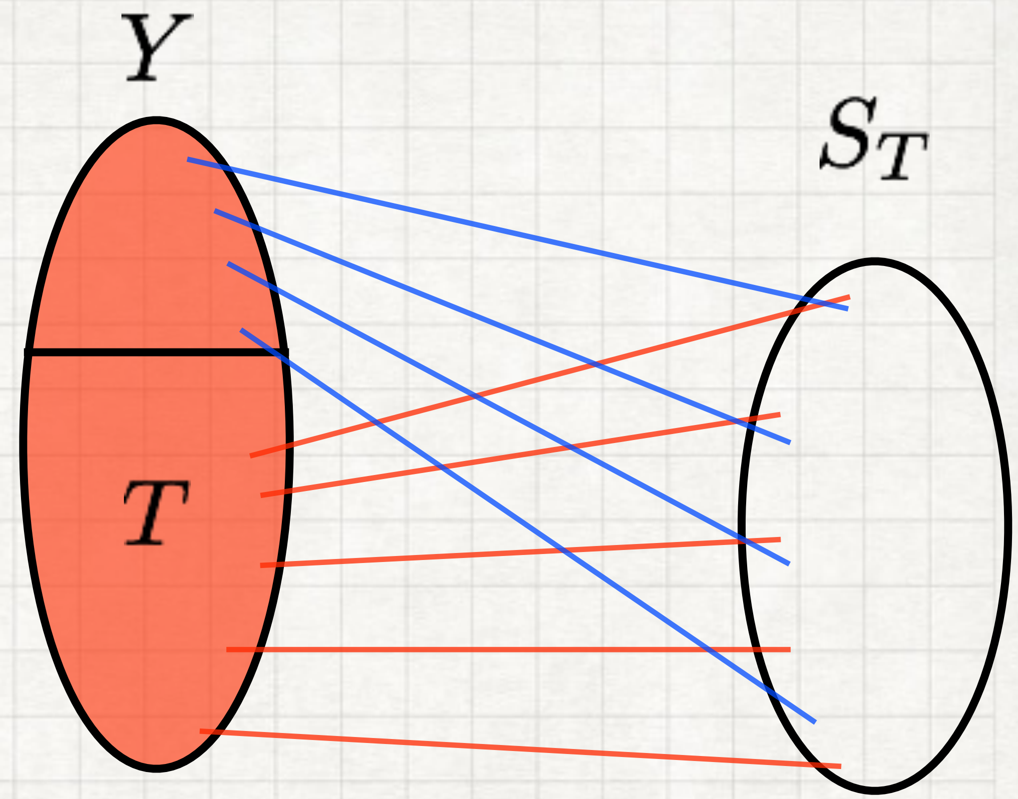
Then all inner products become at most $\frac{-\beta^2}{1-\beta^2} + o(1)$, so by Lemma 1

$$|X| \leq \frac{1-\beta^2}{\beta^2} + o(1) + 1 = \frac{1}{\beta^2} + o(1).$$

□

Suppose $|Y|/2 \leq |T| < |Y|$.



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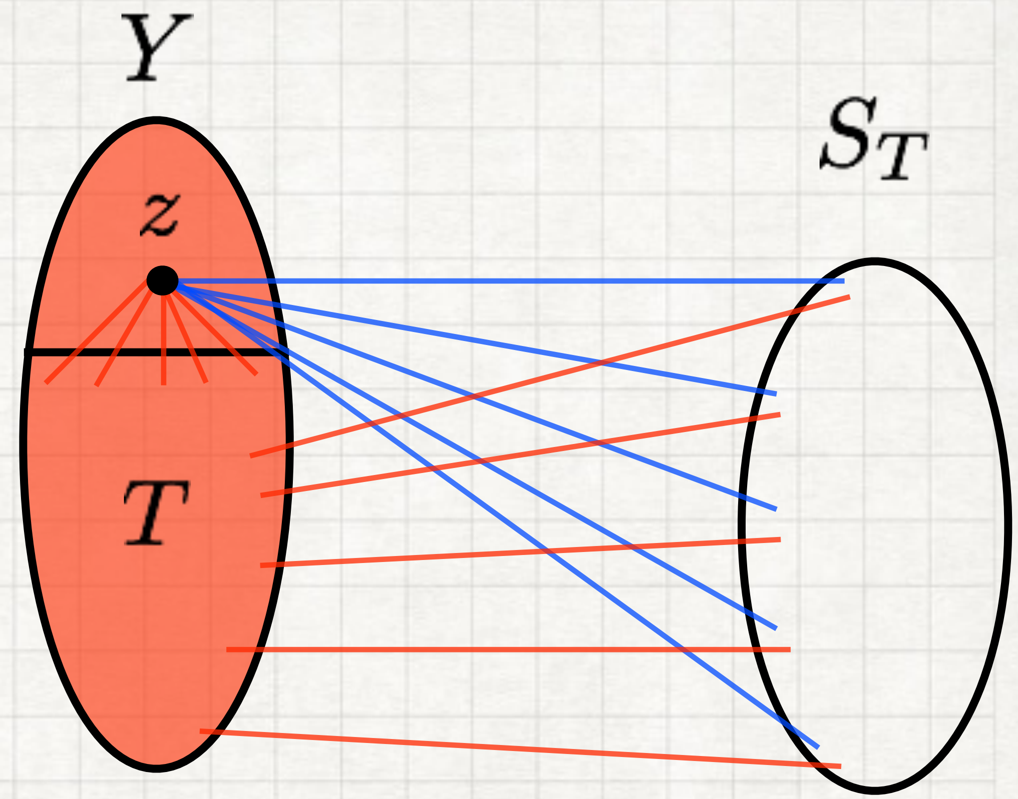

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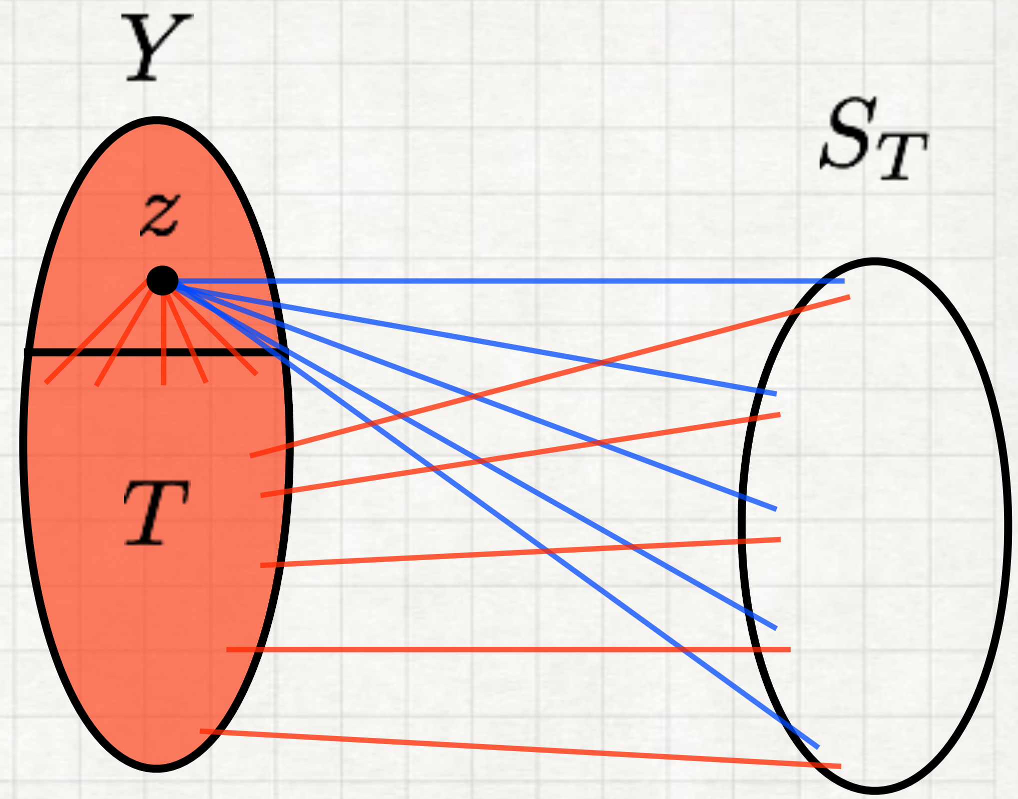
Thus we have that $\sum_{|T|<Y} |S_T| \leq 2^{|Y|}(1/\beta^2 + o(1))$ which we can make $o(d)$, by having $|Y| \to \infty$ slowly enough.



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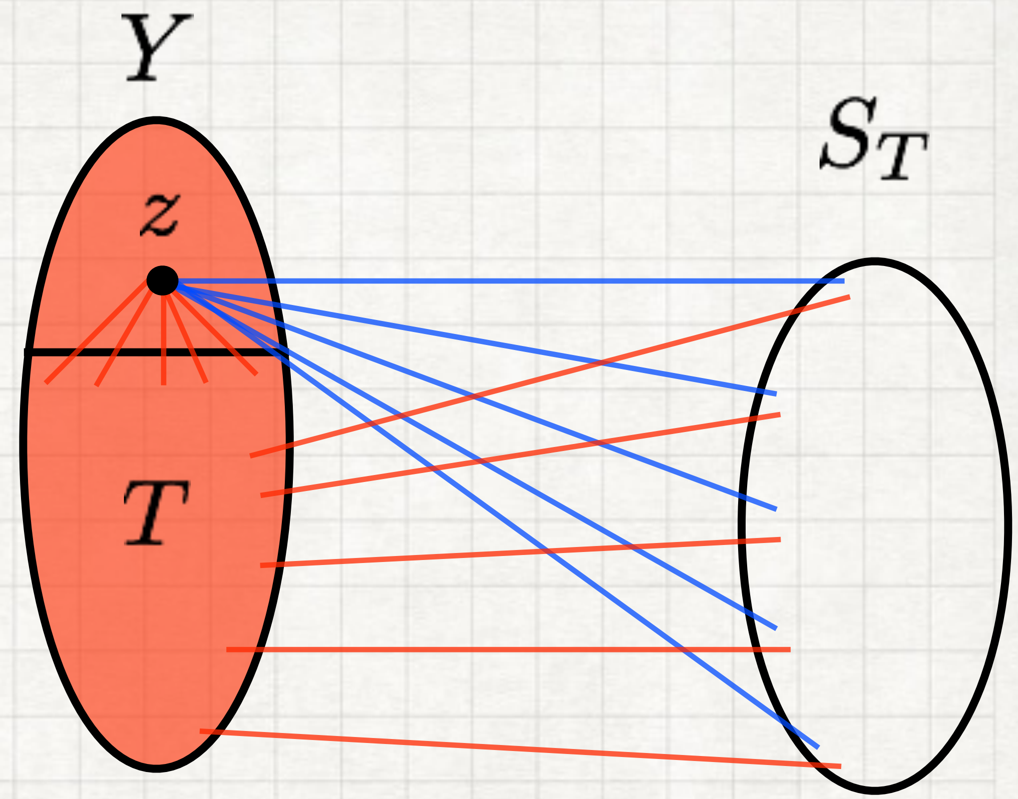

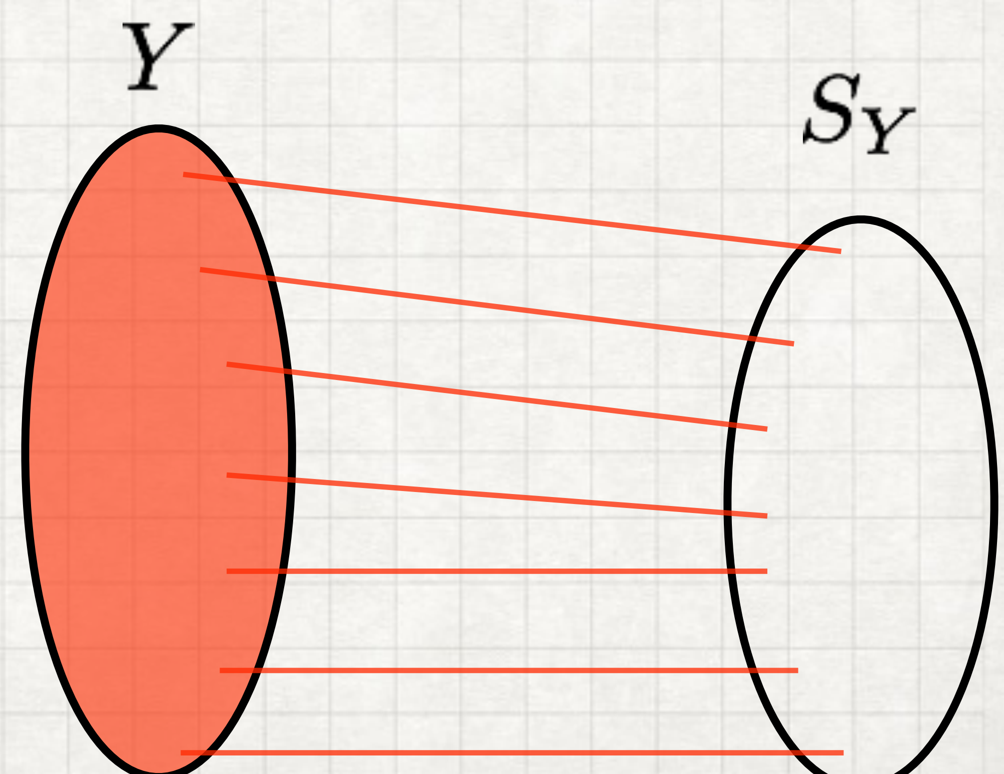

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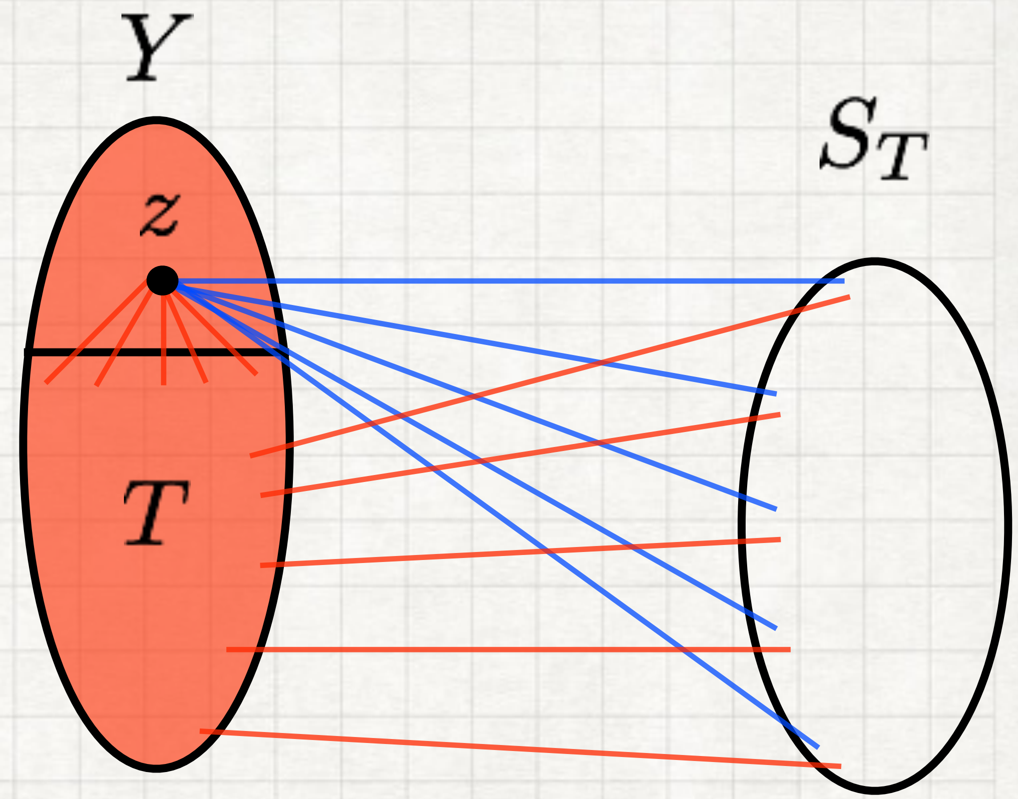

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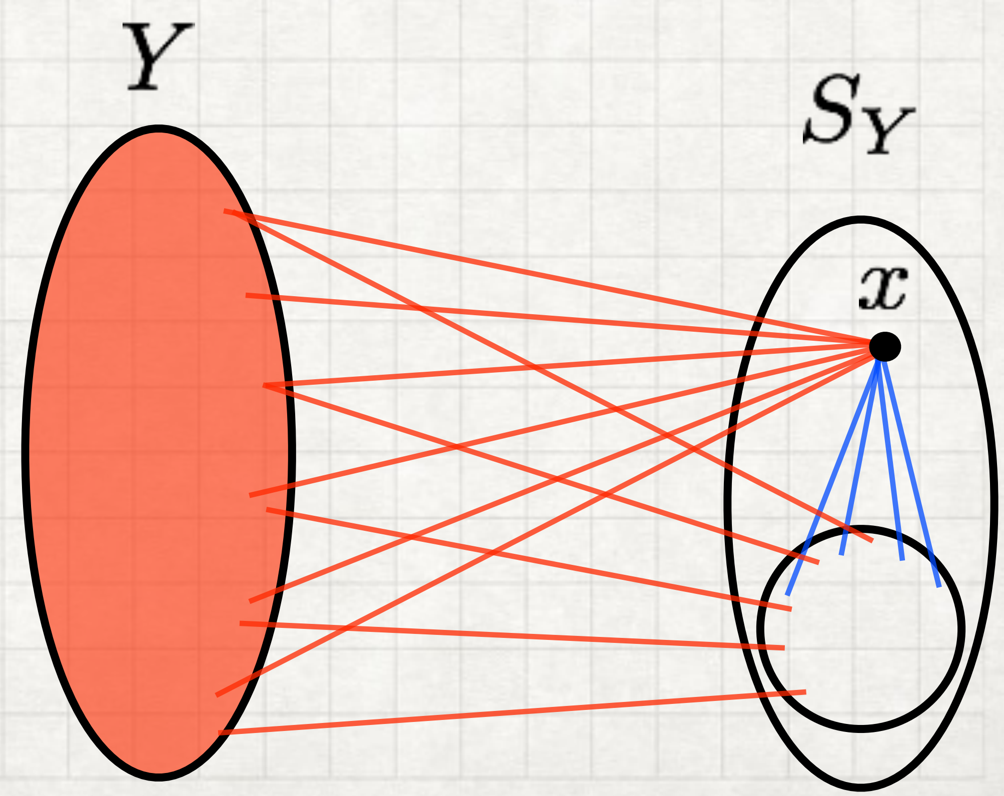
For any $x \in S_Y$, if we apply Lemma 2 to Y, x and the blue neighborhood of x, we see that the blue degree of x is at most $1/\beta^2 + o(1)$.



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Theorem Complete!

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