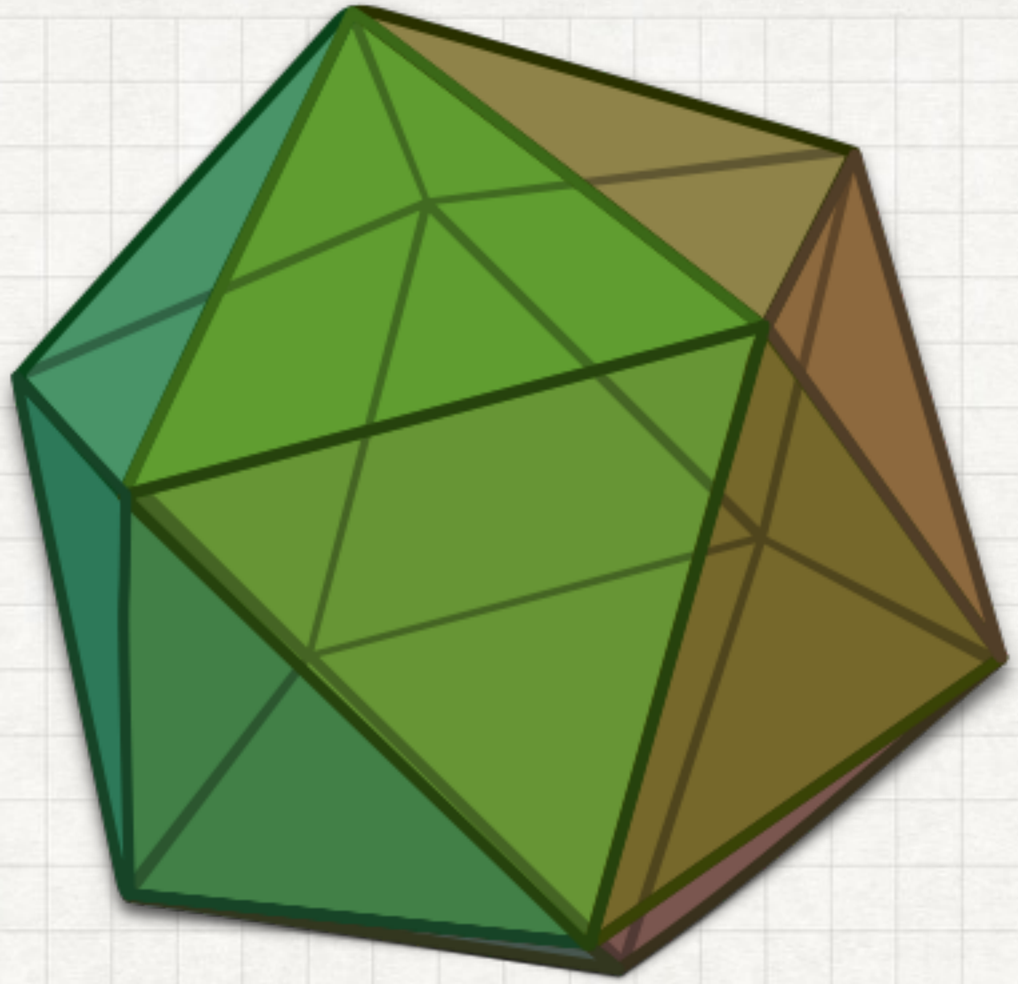

EQUIANGULAR LINES AND SPHERICAL CODES

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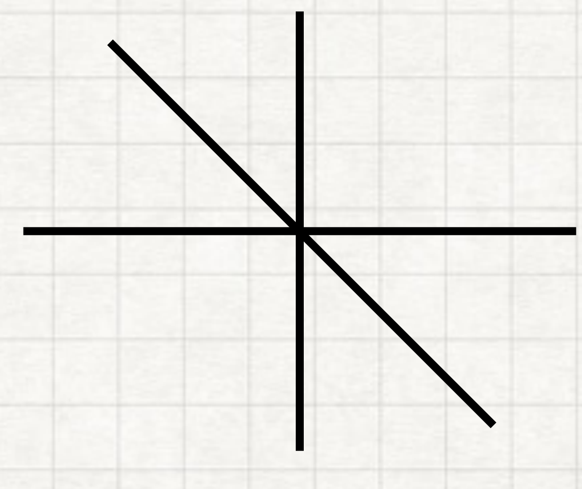
Definition: A set of lines passing through the origin is called **equiangular**, if every pair of lines make the same angle.

Not an example!



Question: What is the maximum number of equiangular lines in $\mathbb{R}^d$?

For d = 2, 3 Greeks?

Earliest work:

Haantjes, Seidel 47-48

Blumenthal 49

Van Lint, Seidel 66

Lemmens, Seidel 73

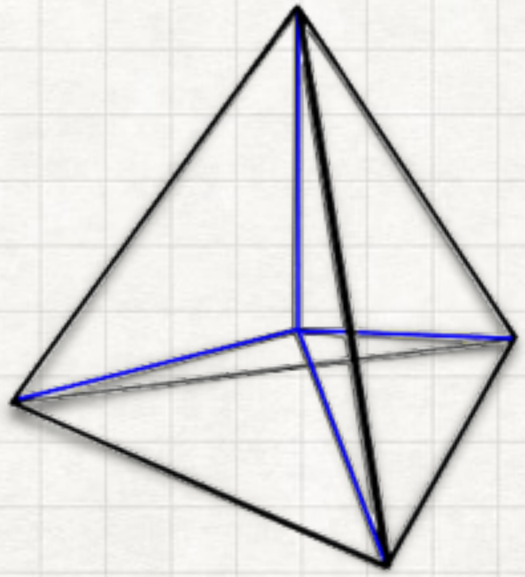
…

Examples

d = 2: Triangle

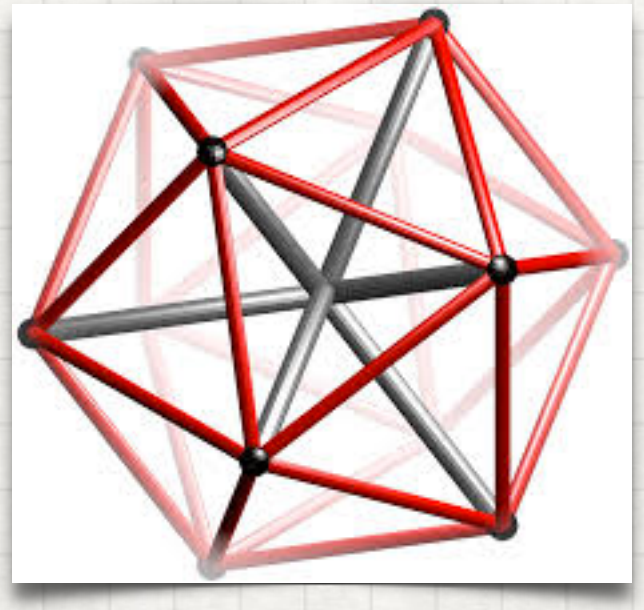
3 lines

In general, d-simplex gives d+1 lines:



d = 3: Icosahedron

6 lines



d = 7:

28 lines

Take all 28 permutations of the vector (3,3,-1,-1,-1,-1,-1,-1).

Theorem (Gerzon 73): The number of equiangular lines in $\mathbb{R}^d$ is at most $\binom{d+1}{2}$.

Proof: Let $x_1, \ldots, x_n$ be unit vectors along the given lines. Then $x_i \cdot x_j = \pm\alpha$ for some $0 \le \alpha < 1$.

Consider the matrices $x_1 x_1^\intercal, \ldots, x_n x_n^\intercal$. They live in the $\binom{d+1}{2}$-dimensional space of symmetric matrices and have the same non-negative inner product:

$$(x_i x_i^\intercal) \cdot (x_j x_j^\intercal) = \mathrm{tr}(x_i x_i^\intercal x_j x_j^\intercal) = (x_i^\intercal x_j)^2 = \begin{cases} 1 & i = j \\ \alpha^2 & i \neq j. \end{cases}$$

Hence they are linearly independent. $\square$

Question: Can we have $\Omega(d^2)$ lines in $\mathbb{R}^d$?

Theorem (de Caen '00 / Jedwab, Wiebe '15 / Greaves, et al. '15):
There exist $\Omega(d^2)$ equiangular lines in $\mathbb{R}^d$.

Remark: These constructions all have an angle of $\Theta\left(\frac{1}{\sqrt{d}}\right) \to 0.$

Question (Lemmens, Seidel 73):
What if the angle is fixed and d tends to infinity?

Theorem (Bukh '15): For fixed α and sufficiently large d, there are at most $2^{O(\alpha^{-2})}d$ equiangular lines.

Theorem (B., Dräxler, Keevash, Sudakov): For fixed α and sufficiently large d, the maximum number of equiangular lines in $\mathbb{R}^d$ is

$$\begin{cases} = 2d-2 & \text{if } \alpha \text{ is } 1/3 \\ \leq 1.93d & \text{otherwise.} \end{cases}$$

Construction of $2d-2$ lines with $\alpha = 1/3$

Definition: For any vectors $x_1, \ldots, x_n \in \mathbb{R}^d$, **the Gram** matrix A is defined by $A_{i,j} = x_i \cdot x_j$. It is $n \times n$, symmetric, positive semidefinite and has rank at most d.

Actually, these conditions are also sufficient, so…

$$\underbrace{\overbrace{\begin{pmatrix} \boxed{\begin{matrix} 1 & -1/3 \\ -1/3 & 1 \end{matrix}} & & 1/3 \\ & \ddots & \\ 1/3 & & \boxed{\begin{matrix} 1 & -1/3 \\ -1/3 & 1 \end{matrix}} \end{pmatrix}}^{2d-2}}_{2d-2 \text{ rows}}$$

eigenvalue	multiplicity
$\frac{2}{3}(d-1)$	1
$4/3$	$d-1$
0	$d-2$

Theorem (B., Dräxler, Keevash, Sudakov): For any fixed α and d sufficiently large, the number of equiangular lines in $\mathbb{R}^d$ with angle α is at most $(2+o(1))d$.

Definition: Call the edge $\{x_i, x_j\}$ red if $x_i \cdot x_j = +\alpha$ and call it blue if $x_i \cdot x_j = -\alpha$. So we get a red-blue edge colored complete graph G on n vertices.

Lemma 1: For $\beta > 0$, if $x_1, \ldots, x_n \in \mathbb{R}^d$ are unit vectors with $x_i \cdot x_j \leq -\beta$, then $n \leq 1/\beta + 1$.

Proof: $0 \leq \| \sum_{i=1}^n x_i \|^2 \leq n - n(n-1)\beta$. $\square$

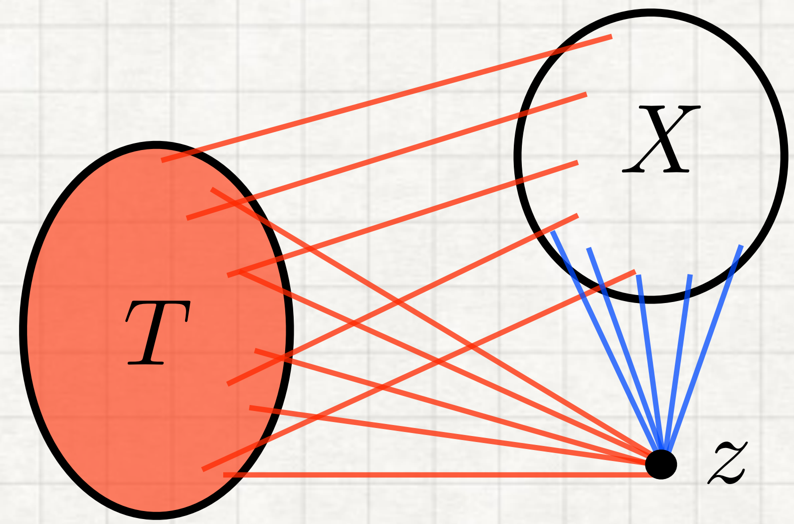
So our graph has no blue clique of size larger than $1/\alpha + 1$.

Thus by Ramsey's theorem it has a large red clique Y. Note that we can take $|Y| \to \infty$ as slowly as we need.

Orthogonal projection

Lemma 2: If T is a red clique with $|T| \to \infty$ and X, z are such that all edges from T to $X \cup \{z\}$ are red and all edges from z to X are blue, then

$$|X| \leq \frac{1}{\beta^2} + o(1) \quad \text{where} \quad \beta := \frac{2\alpha}{1-\alpha}.$$

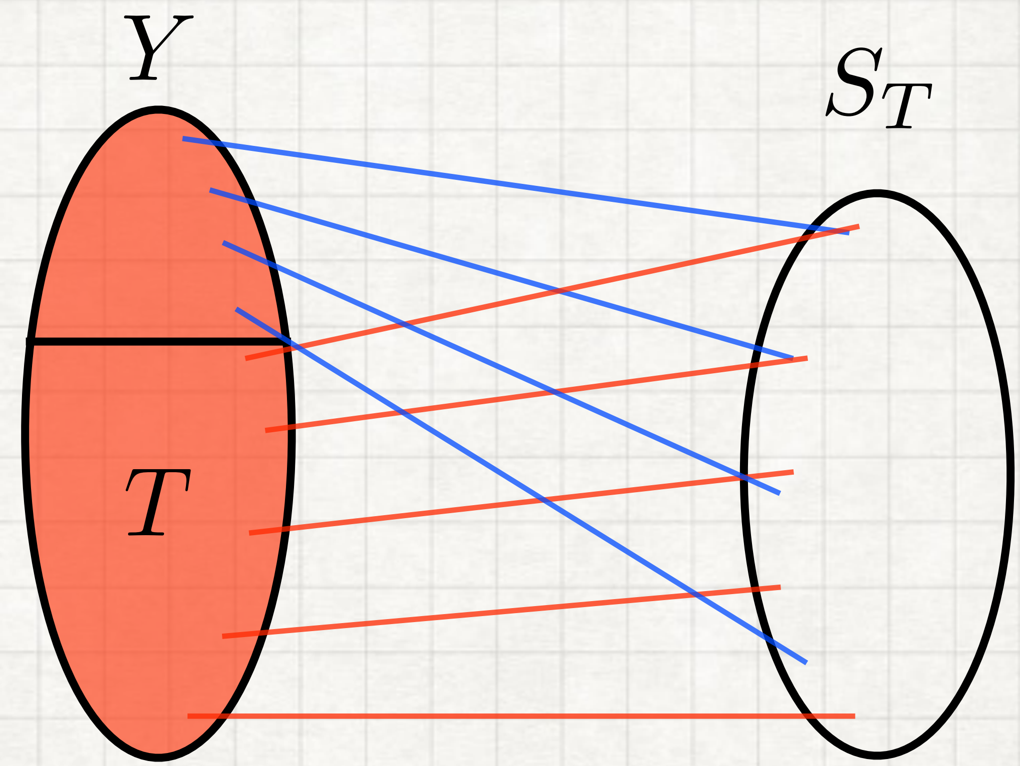


Proof: Project X onto the orthogonal complement of the span of $T \cup \{z\}$ and normalize.
Then all inner products become at most $\frac{-\beta^2}{1-\beta^2} + o(1)$, so by Lemma 1

$$|X| \leq \frac{1-\beta^2}{\beta^2} + o(1) + 1 = \frac{1}{\beta^2} + o(1).$$

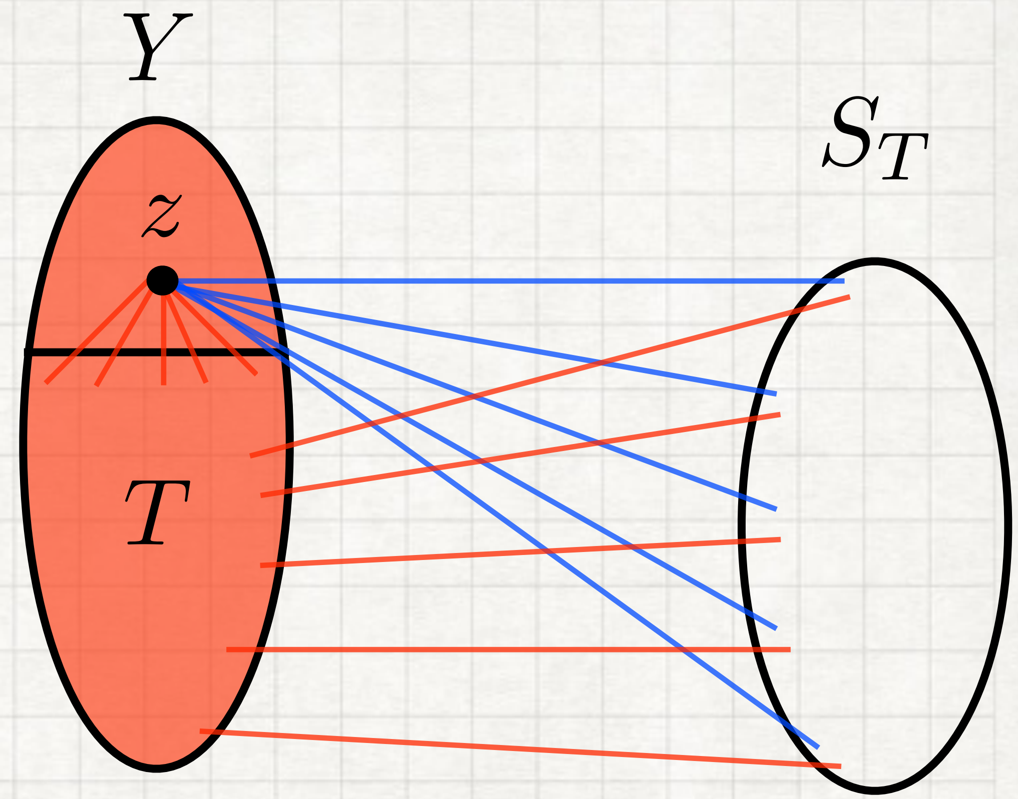
□

Strategy: Show that most of the remaining vertices connect to Y entirely via red edges.



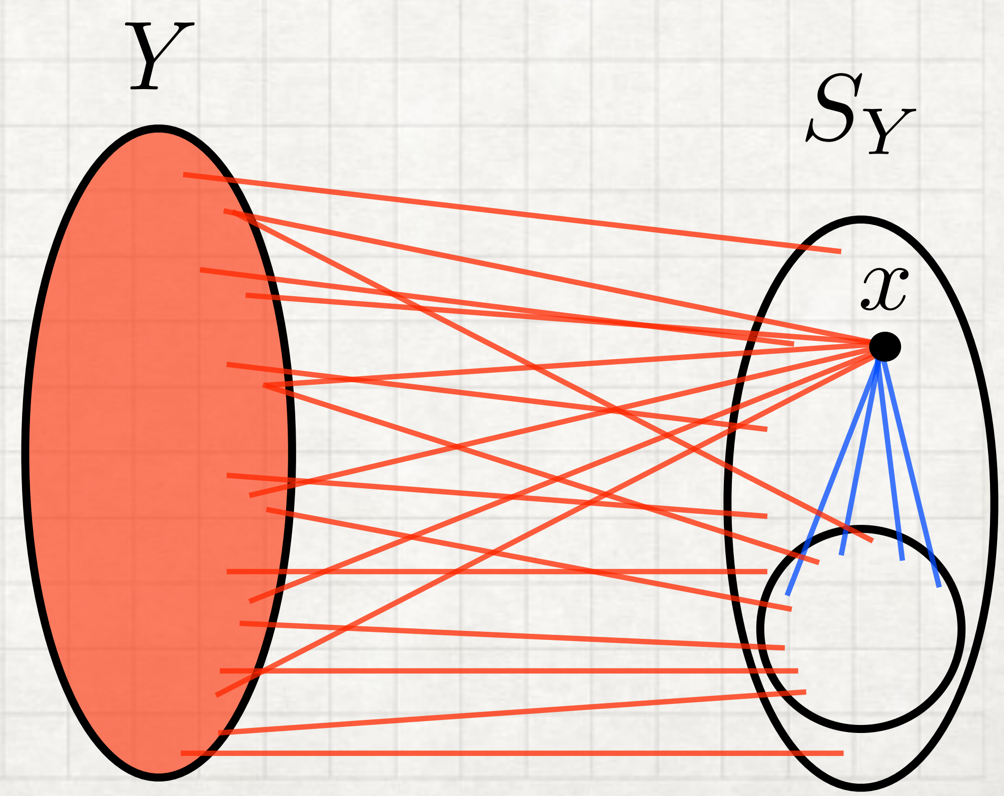
Definition: For any $T \subseteq Y$, define S_T to be those $x \in G \backslash Y$ such that $\{x, y\}$ is red for all $y \in T$, and blue for all $y \in Y \backslash T$.

Choose some $z \in Y \setminus T$, and apply Lemma 2 to T, S_T, z, to conclude that $|S_T| \leq 1/\beta^2 + o(1)$.



Thus we have that $\sum_{|T|<Y} |S_T| \leq 2^{|Y|}(1/\beta^2 + o(1))$ which we can make $o(d)$, by having $|Y| \to \infty$ slowly enough.

So it remains to bound S_Y.



For any $x \in S_Y$, if we apply Lemma 2 to Y, x and the blue neighborhood of x, we see that the blue degree of x is at most $1/\beta^2 + o(1)$.

Now project S_Y onto the orthogonal complement of Y. Then for any red edge, the inner product becomes $\varepsilon = o(1)$ and for any blue edge it becomes $-\beta + o(1)$.

So the Gram matrix A of these vectors looks like

Every row has at most $1/\beta^2 + o(1)$ entries that are $-\beta + o(1)$.

$$\begin{pmatrix} 1 & & & \varepsilon, -\beta + o(1) \\ & 1 & & \\ & & \ddots & \\ \varepsilon, -\beta + o(1) & & & 1 \end{pmatrix}$$

rank at most d

dimension $m = |S_Y|$

Thus if J is the all 1 matrix, then $M = A - \varepsilon J$ looks like

$$\begin{pmatrix} 1 - o(1) & & & 0, -\beta + o(1) \\ & 1 - o(1) & & \\ & & \ddots & \\ 0, -\beta + o(1) & & & 1 - o(1) \end{pmatrix}$$

and the rank r of M is at most $\mathrm{rk}(A) + \mathrm{rk}(-\epsilon J) \leq d + 1$.

Lemma 3 (Schnirelmann 30 / Bellman 60 / Alon '09… **):**
For any symmetric matrix M with rank r, $\mathrm{tr}(M)^2 \leq r\,\mathrm{tr}(M^2)$.

Now we compute

$$\mathrm{tr}(M) = m(1 - o(1))$$

$$\mathrm{tr}(M^2) = \sum_{i,j} (M_{i,j})^2 \leq m\left(1 + \left(\frac{1}{\beta^2} + o(1)\right)(-\beta + o(1))^2\right)$$
$$= m(2 + o(1)).$$

Thus Lemma 3 gives $(m(1 - o(1)))^2 \leq rm(2 + o(1))$,

which implies $m \leq r(2 + o(1))$.

Theorem Complete!