GRAPH LAPLACIANS AND EQUIOVERLAPPING VECTORS

Igor Balla
Based on joint work with Filip Kučerák and Xichao Shu

A set of lines passing through the origin is called **equiangular**, if every pair of lines make the same angle.

Connections:

- Elliptic geometry
- Frame theory
- Theory of polytopes
- Spectral graph theory
- Algebraic number theory
- Quantum information theory

Earliest work:

Haantjes, Seidel 47-48

Blumenthal 49

Van Lint, Seidel 66

Lemmens, Seidel 73

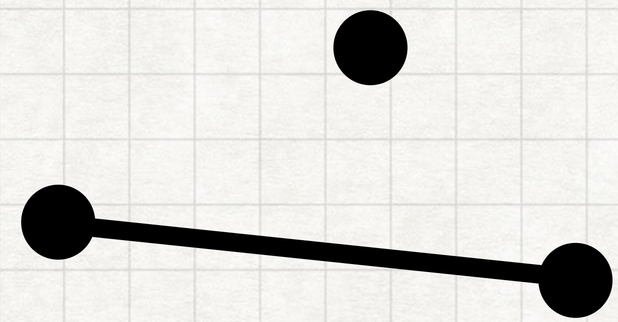
…

Given a family of equiangular lines, if we choose a unit vector along each line, then all pairwise inner products between such vectors will be $\pm\alpha$. Such a collection of vectors is called a **spherical $\{\pm\alpha\}$-code.**

Problem: Determine the maximum number size of a spherical $\{\pm\alpha\}$-code in $\mathbb{R}^r$.

Absolute bound(Gerzon 73): The maximum is at most $\binom{r+1}{2}$.

Given a spherical $\{\pm\alpha\}$-code, we can associate it with a graph where the vertices are the vectors of the code and we put an edge between a pair of vectors u, v if and only if $\langle u, v\rangle = -\alpha$.



Given a spherical $\{\pm\alpha\}$-code, we can consider the corresponding **Gram matrix** whose entries consist of all pairwise inner products among the vectors.

$$M = \begin{pmatrix} 1 & \alpha & \alpha \\ \alpha & 1 & -\alpha \\ \alpha & -\alpha & 1 \end{pmatrix}$$

Also, we can consider the **adjacency matrix** of the corresponding graph.

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$M = (1-\alpha)I + \alpha J - 2\alpha A$$

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This equation also allows us to go from a **regular** graph to a corresponding spherical $\{\pm\alpha\}$-code:

Given a d-regular graph on n vertices whose adjacency matrix has second largest eigenvalue λ_2, let $\alpha = 1/(2\lambda_2 + 1)$ and define M according to the equation.

We just need to ensure that M is positive semidefinite!

Necessary condition: $d - \lambda_2 \leq n/2$

spectral gap/algebraic connectivity

If $d - \lambda_2 < n/2$ then $r = \mathrm{rank}(M) = n - \mathrm{mult}_A(\lambda_2)$, so there exists a spherical $\{\pm\alpha\}$-code of size n in $\mathbb{R}^r$.

Consequences

Given a regular graph on n vertices with second eigenvalue λ_2,

Absolute bound: $\mathrm{mult}_A(\lambda_2) \le n - \frac{\sqrt{8n+1}-1}{2}$.

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Matrix projection(Used in in the context of obtaining better bounds on spherical $\{\pm\alpha\}$-codes when $\alpha \ge \frac{1}{\sqrt{r}}$): For a d-regular graph such that $d - \lambda_2 < (1-\varepsilon)n/2$, we have

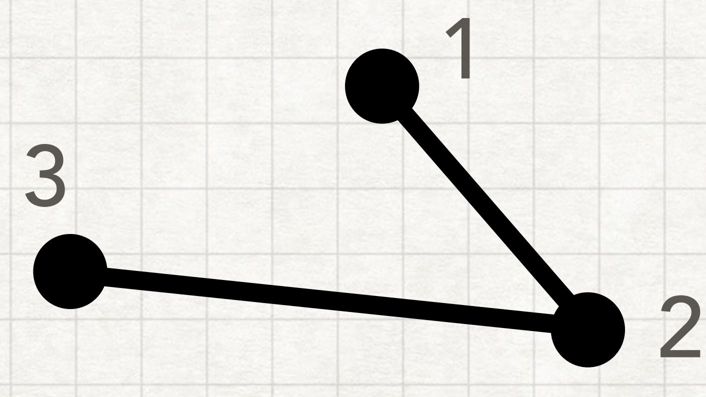
$$\lambda_2 \ge \Omega(d^{1/3}).$$

Generalization

What if instead of the adjacency matrix, we consider the **Laplacian matrix**?

Ex:

$$L = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$



Note that 0 is always the smallest eigenvalue with the all ones vector as a corresponding eigenvector.

Let μ be the second smallest eigenvalue of L.

spectral gap/algebraic connectivity

If a graph is d-regular then $L = dI - A$, so $\mu = d - \lambda_2$.

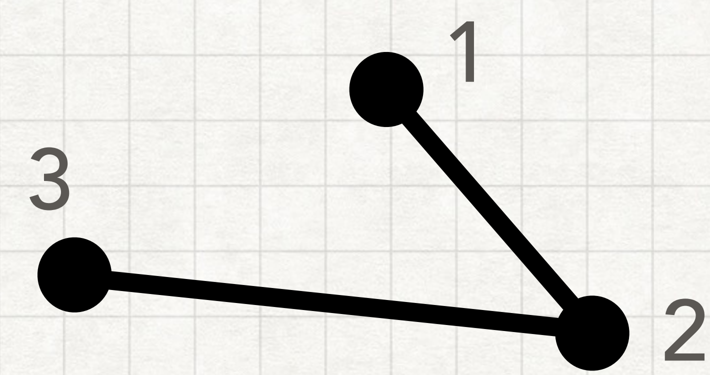
Define $M = J + 2L - 2\mu I$.

Generalization

$$M = J + 2L - 2\mu I$$

Ex:

$$L = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$



$$\mu = 1 \qquad M = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{pmatrix} \qquad |M_{i,j}| = 1 \text{ for all } i \neq j$$

If $\mu < n/2$ then M is positive semidefinite and $r = \text{rank}(M) = n - \text{mult}_L(\mu)$, so M is the Gram matrix of n vectors in $\mathbb{R}^r$!

These vectors are **equioverlapping**, i.e. the absolute value of the inner product of any pair is the same.

Equioverlapping vectors are a generalization of spherical $\{\pm\alpha\}$-codes, where the lengths of the vectors are allowed to vary.

Very recently, equioverlapping measurements have been studied in quantum information theory, see papers by Feng, Lou, Zhao, Gun!

By rescaling the vectors, we will assume WLOG that all pairwise inner products are ± 1.

Problem: Determine the maximum size of a list of equioverlapping vectors in $\mathbb{R}^r$ **none of which have unit length**.

Generalization of the absolute bound: The maximum number is at most $\binom{r+1}{2}$.

Consequences for Laplacians

$$M = J + 2L - 2\mu I$$

Given a graph on n vertices whose Laplacian has second eigenvalue μ and min degree δ such that $\delta > \mu$ and $\mu < n/2$

Absolute bound*: $\mathrm{mult}_L(\mu) \leq n - \frac{\sqrt{8n+9}-1}{2}$.

Matrix projection: If the graph has average degree $\overline{d}$ and max degree Δ such that $\mu \leq (1-\varepsilon)n/2$ and $(\overline{d}-\mu)(\delta-\mu)^2 \geq \varepsilon(\Delta-\mu)$ then

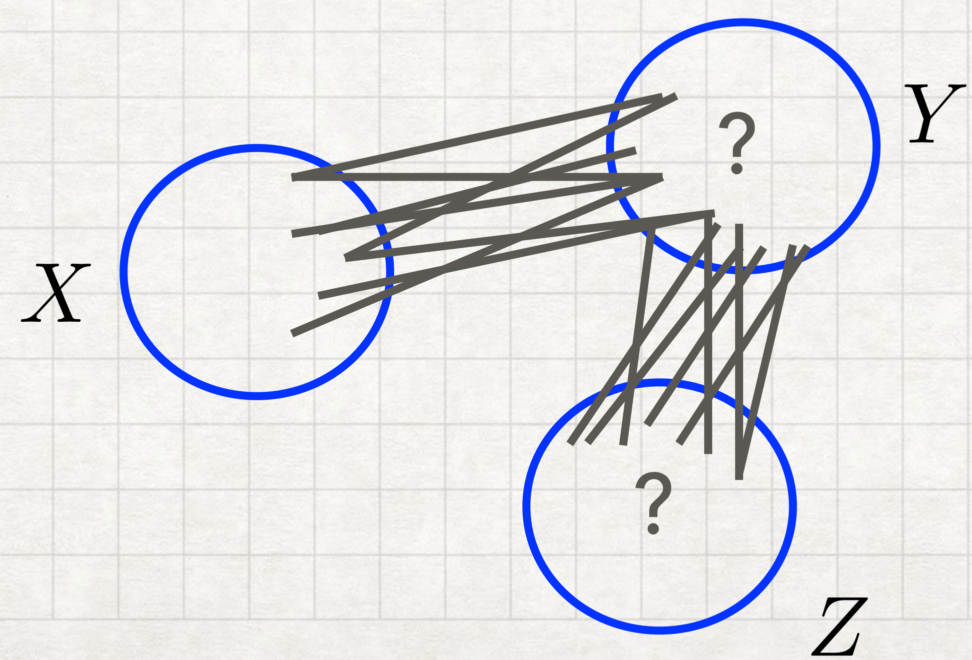
$$\overline{d} - \mu \geq \Omega(\mu^{1/3}).$$

Further questions we answered

Let δ be the minimum degree of the graph. Fiedler (implicitly) showed that $\delta \geq \mu$ whenever the graph is not complete.

Question: What do the graphs that satisfy $\delta = \mu$ look like?

Answer: They must be the disjoint union of three sets of vertices X, Y, Z such that X is a nonempty independent set, $|Y| = \delta$, all edges between X and Y are present, all edges between Y and Z are present, and no edges between X and Z are present.



Further questions we answered

Question: When do equioverlapping vectors come from a Laplacian matrix as we have done?

Answer: Whenever the Gram matrix of the n vectors has the all ones vector as an eigenvector with corresponding eigenvalue λ satisfying $\lambda \leq n$.

Further questions we didn't answer

Question: Can one construct $\binom{r+1}{2}$ equioverlapping vectors (none of unit length) in $\mathbb{R}^r$ such that the vectors have different lengths?

Furthermore, is there such a construction coming from the Laplacian of some graph?

Question: Given n equioverlapping vectors (none of unit length) in $\mathbb{R}^r$ such that all their lengths are bounded, i.e. $O(1)$, is it true that $n \leq O(r)$ for r sufficiently large?

Question: Can one construct r^2 equioverlapping vectors (none of unit length) in $\mathbb{C}^r$ such that the vectors have different lengths?


thank you
danke
謝謝
ngiyabonga
teşekkür ederim
gracias
рахмат
Баярлалаа
спасибо
mersi
kia ora
barka
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dank je
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shukriya
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감사합니다
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תודה רבה