THE MINRANK OF RANDOM GRAPHS OVER ARBITRARY FIELDS

By: Igor Balla

Joint work with: Noga Alon, Lior Gishboliner, Adva Mond, Frank Mousset

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Theorem (Grimmett and McDiarmid 75): For p fixed, the random graph $G \sim G(n, p)$ has chromatic number $\chi(G) = \Theta(n / \log n)$ with high probability.

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For finite fields $\mathbb{F}$ with $|\mathbb{F}| \leq n^{O(1)}$, this result was already proven recently by Golovnev, Regev, and Weinstein.

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By taking a union bound over all matrices of rank at most k, we have

$$\underset{G \sim G(n,p)}{\mathrm{P}}[\mathrm{mr}(G) \leq k] \leq \sum_{M \in \mathbb{F}^{n \times n} : \mathrm{rk}(M) \leq k} \mathrm{P}[M \text{ represents } G]$$

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On the other hand, any independent set in G corresponds to a full rank submatrix of M, and so must have size at most $\mathrm{rk}(M)$. $\square$

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If $M \in \mathbb{F}^{n \times n}$ is a matrix of rank k with sparsity s which is sufficiently **"nice"**, then each row and column will have $\approx s/n$ nonzero entries and so the k linearly independent rows/columns determining M will each have $\approx ks/n$ nonzero entries.

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Lemma (Golovnev, Regev, Weinstein 17)**:** Every rank k matrix $M\in\mathbb{F}^{n\times n}$ has a **"nice"** $n'\times n'$ principal submatrix of rank k' such that $k'/n'\le k/n$.

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Thus in our case we have $m = n^2$, $N = 2kn$, and $d = 2$ so we obtain

$$|\mathscr{Q}| \leq \binom{2n^2 + 2kn}{2kn} \leq (4n^2)^{2kn} .$$

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Done!