THE MINRANK OF RANDOM GRAPHS OVER ARBITRARY FIELDS

By: Igor Balla
Joint work with: Noga Alon, Lior Gishboliner, Adva Mond, Frank Mousset

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Theorem (Grimmett and McDiarmid 75): For p fixed, the random graph $G \sim G(n, p)$ has chromatic number $\chi(G) = \Theta(n/\log n)$ with high probability.

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For finite fields $\mathbb{F}$ with $|\mathbb{F}| \leq n^{O(1)}$, this result was already proven recently by Golovnev, Regev, and Weinstein.

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By taking a union bound over all matrices of rank at most k, we have

$$\mathrm{P}[\mathrm{mr}(G) \leq k] \leq \sum_{M \in \mathbb{F}^{n \times n} :\, \mathrm{rk}(M) \leq k} \mathrm{P}[M \text{ represents } G]$$

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On the other hand, any independent set in G corresponds to a full rank submatrix of M, and so must have size at most $\mathrm{rk}(M)$. $\square$

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If $M \in \mathbb{F}^{n \times n}$ is a matrix of rank k with sparsity s which is sufficiently **"nice"**, then each row and column will have $\approx s/n$ nonzero entries and so the k linearly independent rows/columns determining M will each have $\approx ks/n$ nonzero entries.

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Lets bound the number of **"nice"** matrices with rank at most k and sparsity s.

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Lemma (Golovnev, Regev, Weinstein 17)**:** Every rank k matrix $M \in \mathbb{F}^{n\times n}$ has a **"nice"** $n' \times n'$ principal submatrix of rank k' such that $k'/n' \le k/n$.

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Thus if we let Q_s denote the set of all zero-patterns of matrices $M \in \mathbb{F}^{n \times n}$ with $\mathrm{rk}(M) \leq k$ and $\mathrm{s}(M) = s$, then

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Theorem (Rónyai, Babai, Ganapathy 01)**:** If $(f_1, \ldots, f_m)$ is a vector of polynomials over a field $\mathbb{F}$ with degree $\leq d$ in N variables, then the number zero-patterns of this vector as the variables range over all possible elements is at most $\binom{md+N}{N}$.

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Thus we have n^2 polynomials of degree $\leq k+1$ in $2ks/n$ variables, so

$$|Q_s| \leq \binom{n}{k}^2 \binom{kn}{ks/n}^2 \binom{n^2(k+1)+2ks/n}{2ks/n}.$$

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Note that all entries of C', v' lie in E.

Cramer's rule: If C'_i is the matrix obtained by replacing the i-th column of C' by v', then $x_i = \det(C'_i)/\det(C')$.

A Better Proof Not Depending on $\mathbb{F}$

Correct Claim: Each entry of $\det(C')M$ is a polynomial of degree $\leq k+1$ in the variables of E.

Proof: Suppose the first k rows and columns of M are linearly independent, and let C be the matrix consisting of these columns.

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Done!