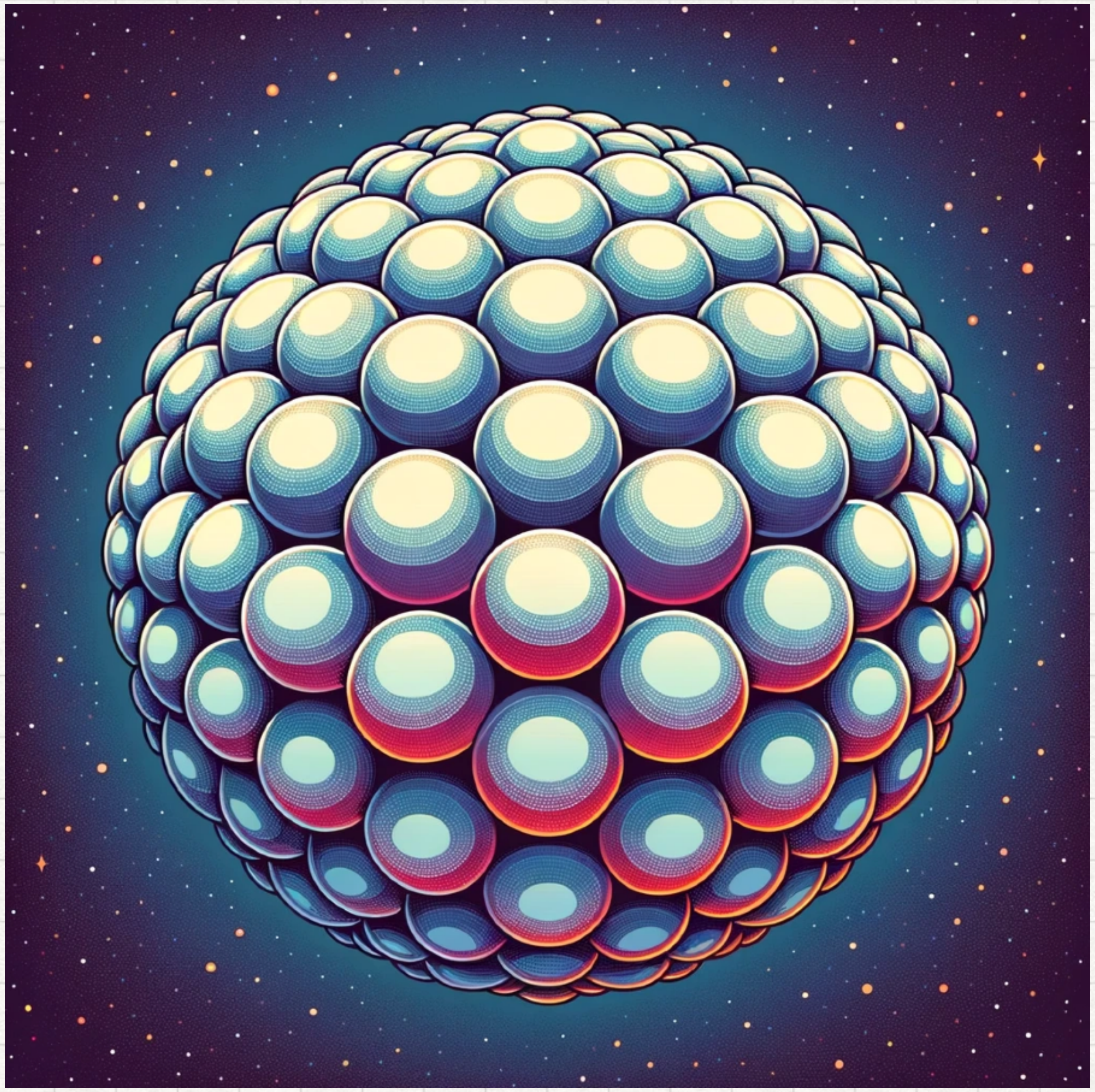


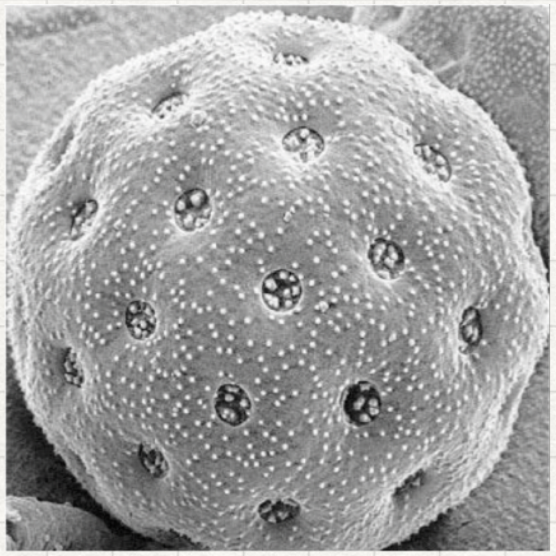
SMALL CODES

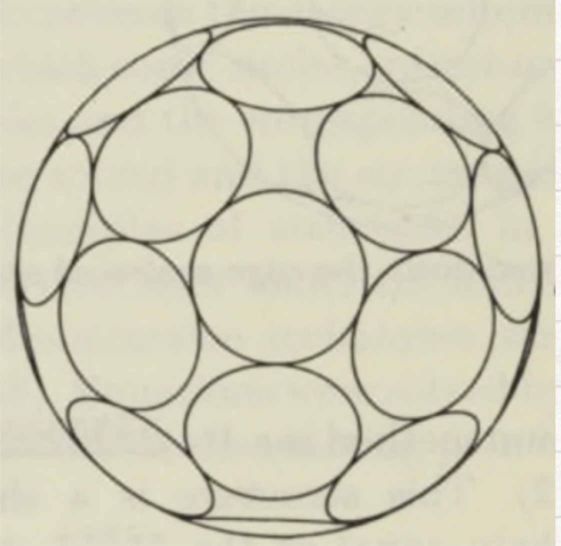
By: Igor Balla

Definition: A collection of vectors $v_1, \ldots, v_n$ on the unit sphere in $\mathbb{R}^d$ with the property that $\langle v_i, v_j \rangle \leq \alpha$ for all $i \neq j$, is called a spherical α-code.

Problem: Determine $M(d, \alpha)$, the maximum size of a spherical α-code in $\mathbb{R}^d$.

Earliest work: By the botanist Tammes in 1930, in the context of studying the arrangements of pores of pollen grains





Equivalent to the question of packing spherical caps of fixed radius on a sphere.

What is known?

Theorem (Rankin 1955):

$$M(d,\alpha) = \begin{cases} \frac{-1}{\alpha} + 1 & \text{if } -1 \le \alpha \le \frac{-1}{d} \\ d+1 & \text{if } \frac{-1}{d} \le \alpha < 0 \\ 2d & \text{if } \alpha = 0 \end{cases}$$

Equivalent formulation: $\rho(d,n) = \min\{\alpha : M(d,\alpha) \ge n\}$, which Rankin's result determines for all $n \le 2d$. In particular, $\rho(d,2d) = 0$.

Question (Bukh and Cox 2020): For a fixed positive k, how does $\rho(d, 2d+k)$ depend on d?

Theorem (Bukh and Cox 2020): If d is sufficiently large relative to k
Then $\Omega\left(\frac{k}{d^2}\right) \le \rho(d, 2d+k) \le O\left(\frac{\sqrt{k}}{d}\right)$.

Theorem (B.): $\rho(d, 2d+k) \ge \Omega\left(\frac{k^{1/3}}{d}\right)$.

New results

Theorem (B.): If $0 \leq \alpha \ll d^{-2/3}$ as $d \to \infty$, then

$$M(d, \alpha) = (2 + o(1))d.$$

Definition: Let $A_2(d, s)$ be the maximum size of a binary code with block length d and minimum Hamming distance s.

Theorem (Plotkin 1960): $A_2(d, d/2) \leq 2d$ with equality if there exists a Hadamard matrix of order d.

Conjecture (Tietäväinen 1980): If $j \ll d^{1/3}$, $A_2(d, d/2 - j) \leq O(d)$.

Corollary (B.): If $j \ll d^{1/3}$, $A_2(d, d/2 - j) \leq (2 + o(1))d$.

Theorem(Sidelnikov 1971): For infinitely many d, there exists $j = \Theta\left(d^{1/3}\right)$ such that $A_2\left(d, d/2 - j\right) \geq \Omega\left(d^{4/3}\right)$.

q-ary codes and set-coloring Ramsey numbers

Definition: Let $R(q; d, s)$ be the minimum number of vertices of a complete graph such that if each of its edges receives s colors out of a universe of d colors, then there exists a set of q vertices all of whose edges share a color.

Fix the threshold $s = \left(1 - \frac{1}{q}\right) d$.

Theorem (Conlon, Fox, He, Mubayi, Suk, Verstraëte 2022 & Conlon, Fox, Pham, Zhao 2023)**:** If $j \ll s$, then

$$A_q\left(d, s - j\right) + 1 \leq R\left(q + 1; d, s - j\right) \leq (1 + o(1))A_q\left(d, s - O(j)\right).$$

Corollary (B.)**:** If $j \ll d^{1/3}$, then $A_q\left(d, s - j\right) \leq (2 + o(1))(q - 1)d$.

Theorem (Mackenzie and Seberry 1988)**:** For all $d \geq q$, $A_q\left(d, s\right) \leq qd$ with equality if d, q are powers of the same prime.