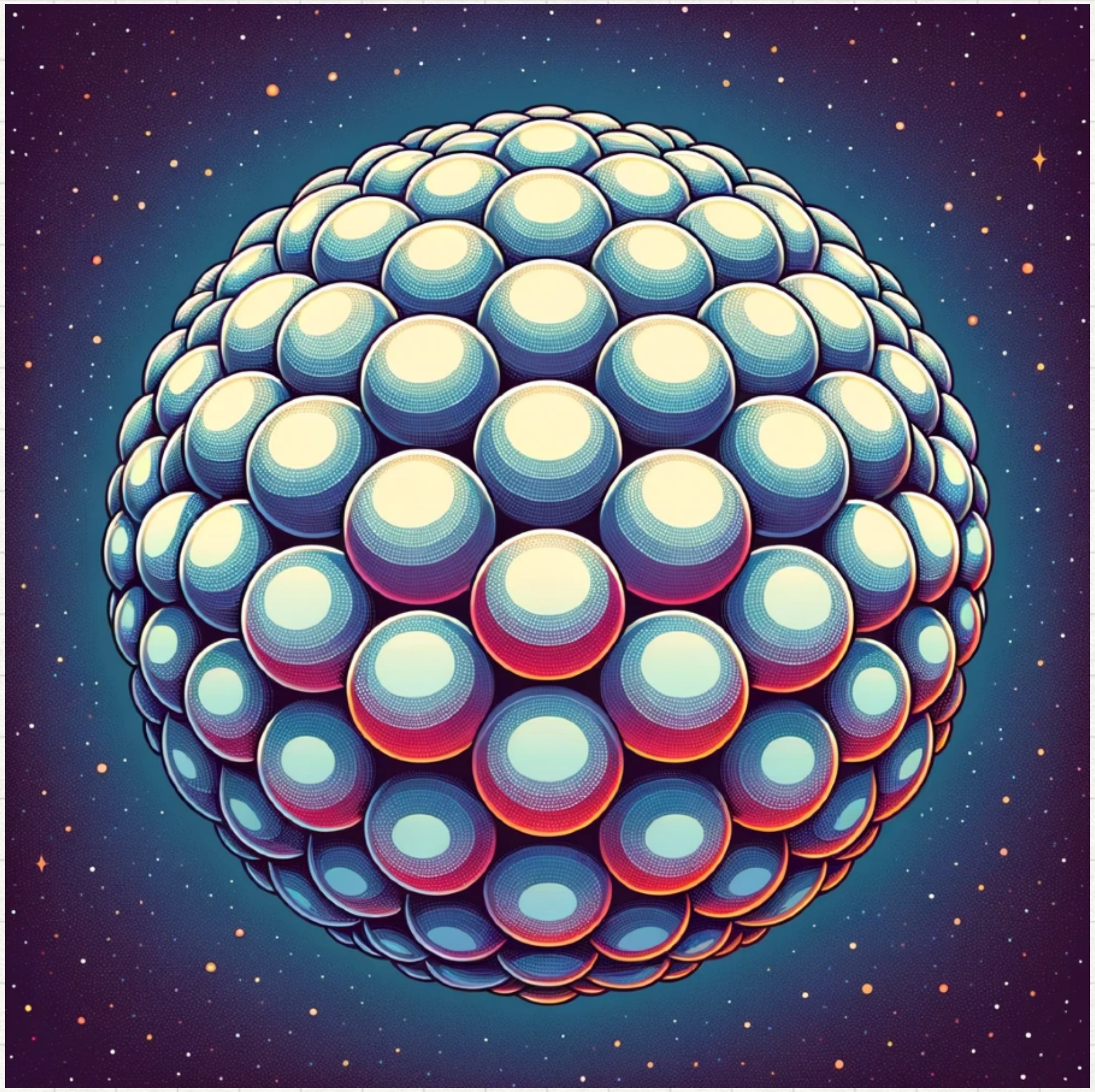


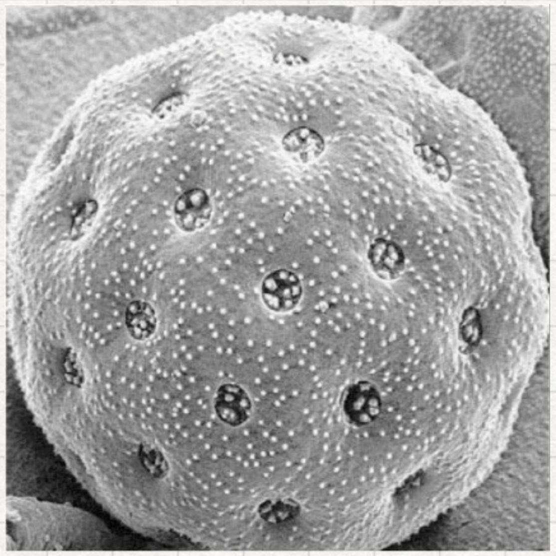
SMALL CODES

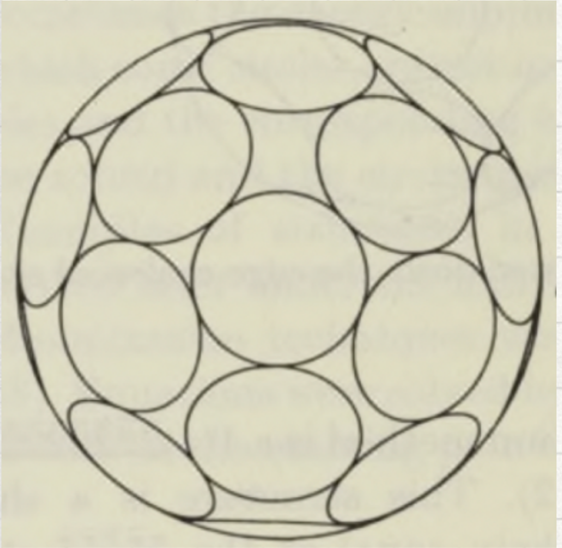
By: Igor Balla

Definition: A collection of vectors $v_1, \ldots, v_n$ on the unit sphere in $\mathbb{R}^d$ with the property that $\langle v_i, v_j \rangle \leq \alpha$ for all $i \neq j$, is called a spherical α-code.

Problem: Determine $M(d, \alpha)$, the maximum size of a spherical α-code in $\mathbb{R}^d$.

Earliest work: By the botanist Tammes in 1930, in the context of studying the arrangements of pores of pollen grains





Equivalent to the question of packing spherical caps of fixed radius on a sphere.

What is known?

Theorem (Rankin 1955):

$$M(d,\alpha) = \begin{cases} \frac{-1}{\alpha} + 1 & \text{if } -1 \le \alpha \le \frac{-1}{d} \\ d+1 & \text{if } \frac{-1}{d} \le \alpha < 0 \\ 2d & \text{if } \alpha = 0 \end{cases}$$

Equivalent formulation: $\rho(d,n) = \min\{\alpha : M(d,\alpha) \ge n\}$, which Rankin's result determines for all $n \le 2d$. In particular, $\rho(d,2d) = 0$.

Question (Bukh and Cox 2020): For a fixed positive k, how does $\rho(d, 2d+k)$ depend on d?

Theorem (Bukh and Cox 2020): If d is sufficiently large relative to k, then $\Omega\left(\frac{k}{d^2}\right) \le \rho(d, 2d+k) \le O\left(\frac{\sqrt{k}}{d}\right)$.

New results

Theorem (B.): $\rho(d, 2d+k) \geq \Omega\left(\frac{k^{1/3}}{d}\right)$.

Corollary (B.): If k is fixed, then $\rho(d, 2d+k) = \Theta(1/d)$ as $d \to \infty$.

Corollary (B.): If $0 \leq \alpha \ll d^{-2/3}$ as $d \to \infty$, then

$$M(d, \alpha) = (2 + o(1))d.$$

Binary codes

Definition: Let $A_q(d,s)$ be the maximum size of a q-ary code with block length d and minimum Hamming distance s.

Theorem (Plotkin 1960): $A_2(d,d/2) \leq 2d$ with equality if there exists a Hadamard matrix of order d.

Conjecture (Tietäväinen 1980): If $j \ll d^{1/3}$, $A_2(d,d/2-j) \leq O(d)$.

Corollary (B.): If $j \ll d^{1/3}$, $A_2(d,d/2-j) \leq (2+o(1))d$.

Corollary (B.): If $j \ll d^{1/3}$, then for infinitely many d, we have $A_2(d,d/2-j) = (2+o(1))d$.

Theorem(Sidelnikov 1971): For infinitely many d, there exists $j = \Theta\left(d^{1/3}\right)$ such that $A_2\left(d,d/2-j\right) \geq \Omega\left(d^{4/3}\right)$.

q-ary codes and set-coloring Ramsey numbers

Definition: Let $R(q;d,s)$ be the minimum number of vertices of a complete graph such that if each of its edges receives s colors out of a universe of d colors, then there exists a set of q vertices all of whose edges share a color.

Fix the threshold $s = \left(1 - \frac{1}{q}\right) d$.

Theorem (Conlon, Fox, He, Mubayi, Suk, Verstraëte 2022 & Conlon, Fox, Pham, Zhao 2023)**:** If $j \ll s$, then

$$A_q\left(d, s-j\right) + 1 \leq R\left(q+1; d, s-j\right) \leq (1+o(1))A_q\left(d, s - O(j)\right).$$

Corollary (B.)**:** If $j \ll d^{1/3}$, then $A_q\left(d, s-j\right) \leq (2+o(1))(q-1)d$.

Theorem (Mackenzie and Seberry 1988)**:** For all $d \geq q$, $A_q\left(d, s\right) \leq qd$ with equality if d, q are powers of the same prime.

Proofs

Theorem (B.): $\rho(d, 2d+k) \geq \Omega\left(\frac{k^{1/3}}{d}\right)$.

Ideas behind the proof: Consider $n = 2d + k$ unit vectors $v_1, \ldots, v_n \in \mathbb{R}^d$ with the property that $\langle v_i, v_j \rangle \leq \alpha$ for all $i \neq j$.

Their Gram matrix $M(i,j) = \langle v_i, v_j \rangle$ has rank at most d and $\mathrm{tr}(M) = n$.

Lets make our life simpler: Imagine that all inner products were positive. Then the off-diagonal entries of M are at most α^2, so it must be close to the identity matrix and hence must have large rank!

Theorem: For any symmetric matrix M, we have

$$\mathrm{tr}(M)^2 \leq \mathrm{rk}(M)\mathrm{tr}(M^2).$$

Proofs

Theorem: For any symmetric matrix M, we have

$$\operatorname{tr}(M)^2 \leq \operatorname{rk}(M)\operatorname{tr}(M^2).$$

But of course life is not so simple. The negative inner products could be **VERY** negative.

Idea: Use the positive semidefiniteness of M ($x^\intercal M x \geq 0$) to gain control over the squares of the negative inner products.

Lemma 1: Fix i and let $\gamma(i) = \sum_{j:\langle v_i, v_j\rangle<0} -\langle v_i, v_j\rangle$. Then

$$\sum_{j:\langle v_i, v_j\rangle<0} \langle v_i, v_j\rangle^2 \leq 1 + \alpha\gamma(i)^2.$$

$$x(j) = \begin{cases} -\langle v_i, v_j\rangle & \text{if } \langle v_i, v_j\rangle < 0 \\ 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

Lemma 2: $\displaystyle\sum_{i=1}^{n} \gamma(i)^2 \leq O\left(\alpha^2 n^3\right).$

$$x(j) = \begin{cases} -\gamma(j) & \text{if } -\gamma(j) \geq 2\alpha n \\ 2\alpha n & \text{otherwise} \end{cases}$$

Unanswered questions

- We showed that if d is sufficiently large relative to k, then $\Omega\left(\frac{k^{1/3}}{d}\right) \le \rho(d, 2d+k) \le O\left(\frac{\sqrt{k}}{d}\right)$. What is the correct dependence on k?

- We showed that if $j \ll d^{1/3}$, then for infinitely many d, we have $qd \le A_q(d, (1-1/q)d - j) \le (2+o(1))(q-1)d$. What is the correct dependence on q for $q \ge 3$?


thank you
рахмат
danke
謝謝
ngiyabonga
teşekkür ederim
Баярлалаа
спасибо
mersi
kia ora
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maururu
köszönöm
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감사합니다
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